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See following page for Table of Contents.

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AUTOMATION AND REMOTE CONTROL

VOLUME 19, NUMBER 2 FEBRUARY 1958

CONTENTS

	PAGE	RUSS. PAGE
On The Stability of Nonlinear Control Systems Described by Fifth and Sixth Order Differential Equations. E. N. Rozenvasser	91	101
On The Theory of One-Relay Systems. V. S. Boiarinov and N. N. Leonov	103	114
Determination of Parameters for Corrective Devices in Linear Servosystems Using Given Generalized Parameters. M. M. Kreimerman	124	135
An Analytical Formulation of the Synthesis Problem of Corrective Devices in Linear Servosystems. V. G. Segalin	137	148
An AC Bridge With Self-Balancing In Two Parameters. V. Iu. Kneller	156	162
Frequency Methods of Remote Control For Distributed Objects. V. A. Il'in and K. P. Kurdiukov	167	174
Letters to the Editor		
A Nonlinear Semiconductor of Resistance Sensitive to Magnetic Fields. G. E. Pikus and O. V. Sorokin	177	187
Chronicle		
The International Federation of Automatic Control (IFAC). A. M. Letov and B. N. Naumov	179	189
Conference on Automatic Control and Computer Technique. I. M. Rusevich	182	191
Errata	186	195

ON THE STABILITY OF NONLINEAR CONTROL SYSTEMS DESCRIBED BY FIFTH AND SIXTH ORDER DIFFERENTIAL EQUATIONS

E. N. Rozenvasser

(Leningrad)

On the basis of a theorem due to Lur'e [1], sufficiency conditions "in the large" are obtained for some control systems described by fifth- and sixth-order differential equations.

Let the control systems under consideration be described by the Equations {I, 1, 19} •

$$\dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha + h_k f(\sigma) \quad (k = 1, 2, \ldots, n),$$
$$\sigma = \sum_{s=1}^{n} j_s h_s \qquad (n = 5, 6). \tag{1}$$

We assume that the characteristic equation of the open System (1)

$$D(\lambda) = |b_{k\alpha} - \lambda\delta_{k\alpha}| = 0 \tag{2}$$

($\delta_{k\alpha}$ is the Kronecker symbol, $\delta_{k\alpha} = \begin{cases} 1 \text{ for } k = \alpha \\ 0 \text{ for } k \neq \alpha \end{cases}$) has roots $\lambda = \lambda_\rho$ ($\rho = 1, 2, \ldots, n$) ($n = 5, 6$) different from zero and with negative real parts.••

The characteristic equation of the linearized System (1), with $f(\sigma) = c\sigma$, may be put in the form

$$\Delta(\mu) = D(\mu) - cM(\mu), \tag{3}$$

where the degree of the polynomial $M(\mu)$ is always less than the degree of the polynomial $D(\mu)$. In the case $n = 6$, it will be assumed that

$$\sum_{s=1}^{6} \frac{\lambda_s M(\lambda_s)}{D'(\lambda_s)} = 0. \tag{4}$$

This relation will hold identically if the degree of $M(\mu)$ does not exceed three. For simplicity we assume that this is indeed the case. Thus, we may assume:

for $n = 5$

$$D(\mu) = -\mu^5 + k_1\mu^4 - k_2\mu^3 + k_3\mu^2 - k_4\mu + k_5, \tag{5}$$
$$M(\mu) = -l_1\mu^4 + l_2\mu^3 - l_3\mu^2 + l_4\mu - l_5; \tag{6}$$

• Formulas and notation taken from Lur'e's book [1] will retain the numeration therein given. The corresponding citations are given within braces, the Roman numeral giving the chapter, the Arabic numerals giving the number of the formula within the chapter.

•• The case will also be considered where $n = 6$, $\lambda_6 = 0$, $r = -\sum j_s h_s = 0$.

for n = 6

$$D(\mu) = \mu^6 - m_1\mu^5 + m_2\mu^4 - m_3\mu^3 + m_4\mu^2 - m_5\mu + m_6, \tag{7}$$

$$M(\mu) = n_1\mu^3 - n_2\mu^2 + n_3\mu - n_4. \tag{8}$$

According to a theorem of Lur'e [1], sufficiency conditions for stability "in the large" for systems (1) can be obtained as necessary and sufficient conditions for the existence of real solutions of a certain system of quadratic equations. This latter system is called the resolving system. In considering the case corresponding to {II.8.16-21}, and taking (4) into account, we may put the resolving systems in the form:

for n = 5

$$\begin{gathered}
-(t_0\sigma_4)^2 + \Gamma_1 = 0, \\
(t_0\sigma_5 + t_1\sigma_4)^2 - 2t_0\sigma_4(t_0\sigma_6 + t_1\sigma_5 + t_2\sigma_4) + \Gamma_3 = 0, \\
-(t_0\sigma_6 + t_1\sigma_5 + t_2\sigma_4)^2 - 2t_0\sigma_4(t_0\sigma_8 + t_1\sigma_7 + t_2\sigma_6 + t_3\sigma_5 + t_4\sigma_4) + \\
+ 2(t_0\sigma_5 + t_1\sigma_4)(t_0\sigma_7 + t_1\sigma_6 + t_2\sigma_5 + t_3\sigma_4) + \Gamma_5 = 0, \\
-(t_4\sigma_{-1})^2 + \Gamma_{-1} = 0, \\
(t_4\sigma_{-2} + t_3\sigma_{-1})^2 - 2t_4\sigma_{-1}(t_4\sigma_{-3} + t_3\sigma_{-2} + t_2\sigma_{-1}) + \Gamma_{-3} = 0;
\end{gathered} \tag{9}$$

for n = 6

$$\begin{gathered}
(t_1\sigma_5)^2 + R_3 = 0, \\
-(t_1\sigma_6 + t_2\sigma_5)^2 + 2t_1\sigma_5(t_1\sigma_7 + t_2\sigma_6 + t_3\sigma_5) + R_5 = 0, \\
-(t_5\sigma_{-1})^2 + R_{-1} = 0, \\
(t_5\sigma_{-2} + t_4\sigma_{-1})^2 - 2t_5\sigma_{-1}(t_5\sigma_{-3} + t_4\sigma_{-2} + t_3\sigma_{-1}) + R_{-3} = 0, \\
-(t_5\sigma_{-3} + t_4\sigma_{-2} + t_3\sigma_{-1})^2 - 2t_5\sigma_{-1}(t_5\sigma_{-5} + t_4\sigma_{-4} + t_3\sigma_{-3} + t_2\sigma_{-2} + t_1\sigma_{-1}) + \\
+ 2(t_5\sigma_{-2} + t_4\sigma_{-1})(t_5\sigma_{-4} + t_4\sigma_{-3} + t_3\sigma_{-2} + t_2\sigma_{-1}) + R_{-5} = 0.
\end{gathered} \tag{10}$$

In Systems (9) and (10) the t_i are the unknowns,

$$\begin{gathered}
\Gamma_1 = -l_1\sigma_5 + l_2\sigma_4, \quad \Gamma_3 = -l_1\sigma_7 + l_2\sigma_6 - l_3\sigma_5 + l_4\sigma_4, \\
\Gamma_5 = -l_1\sigma_9 + l_2\sigma_8 - l_3\sigma_7 + l_4\sigma_6 - l_5\sigma_5, \\
\Gamma_{-1} = -l_5\sigma_{-1}, \quad \Gamma_{-3} = -l_3\sigma_{-1} + l_4\sigma_{-2} - l_5\sigma_{-3},
\end{gathered} \tag{11}$$

$$\begin{gathered}
R_3 = n_1\sigma_6 - n_2\sigma_5, \quad R_5 = n_1\sigma_8 - n_2\sigma_7 + n_3\sigma_6 - n_4\sigma_5, \\
R_{-1} = -n_4\sigma_{-1}, \quad R_{-3} = -n_2\sigma_{-1} + n_3\sigma_{-2} - n_4\sigma_{-3}, \\
R_{-5} = n_1\sigma_{-3} - n_2\sigma_{-3} + n_3\sigma_{-4} - n_4\sigma_{-5}.
\end{gathered} \tag{12}$$

In what follows, we shall assume that Γ_1, Γ_{-1}, R_3 and R_{-1} are all different from zero; otherwise we would be back to the case previously investigated [1]. The constants $\sigma_k (k = 1, 2, \ldots, n+3)$, $\sigma_{-k} (k = 1, 2, 3, 4)$ are computed from the coefficients of (5) and (7) in accordance with {II.8.12-13}, and the quantities σ_{-5} and σ_9, obtained by methods given in [1], equal

$$\sigma_{-5} = \frac{m_2}{m_6^2} - \frac{m_4^2}{m_6^3} - \frac{2m_3m_5}{m_6^3} + \frac{3m_5^2m_4}{m_6^4} - \frac{m_5^4}{m_6^5}, \tag{13}$$

$$\sigma_9 = \sigma_{n+4} = -k_1^5 + 4k_1^3k_2 - 3k_1^2k_3 - 3k_1k_2^2 + 2k_2k_3 + 2k_1k_4 - k_5. \tag{14}$$

For the existence of real solutions of the resolving systems (9) and (10), it is obviously necessary that the following inequalities hold:

for n = 5

$$\Gamma_1 > 0, \ \Gamma_{-1} > 0, \tag{15}$$

for n = 6

$$R_{-2} < 0, \quad R_{-1} > 0. \tag{16}$$

In this case we have for System (9)

$$t_0 = \frac{\varepsilon_1 \sqrt{\Gamma_1}}{\sigma_4}, \quad t_4 = \frac{\varepsilon_1 \sqrt{\Gamma_{-1}}}{\sigma_{-1}}, \quad \varepsilon_1 = \pm 1, \tag{17}$$

and for System (10)

$$t_1 = \frac{\varepsilon_2 \sqrt{-R_2}}{\sigma_{-2}}, \quad t_5 = \frac{\varepsilon_2 \sqrt{R_{-1}}}{\sigma_{-1}}, \quad \varepsilon_2 = \pm 1. \tag{18}$$

If (17) and (18) hold, then systems (9) and (10) may be transformed to the single form:

$$x^2 - \varepsilon_1 z = A, \quad y^2 - \varepsilon_2 z = B, \quad z^2 - xy = C, \tag{19}$$

with the following notation used:

for n = 5

$$x = \frac{t_1}{2\sqrt{2|t_0|\sqrt{|t_0 t_4|}}}, \quad y = \frac{t_3}{2\sqrt{2|t_4|\sqrt{|t_0 t_4|}}}, \quad z = \frac{t_2}{4\sqrt{|t_0 t_4|}}, \tag{20}$$

$$A = \frac{t_0^2(2\sigma_4\sigma_6 - \sigma_5^2) - \Gamma_2}{8\sigma_4^2 |t_0| \sqrt{|t_0 t_4|}}, \quad B = \frac{t_4^2(2\sigma_{-1}\sigma_{-3} - \sigma_{-2}^2) - \Gamma_{-2}}{8\sigma_{-1}^2 |t_4| \sqrt{|t_0 t_4|}},$$

$$C = \frac{-2t_0 t_4 \sigma_4^2 + 8A|t_0|\sqrt{|t_0 t_4|}(2\sigma_4\sigma_6 - \sigma_5^2) - t_0^2(\sigma_6^2 + 2\sigma_4\sigma_8 - 2\sigma_5\sigma_7) + \Gamma_3}{16\sigma_4^2 |t_0 t_4|}, \tag{21}$$

for n = 6

$$x = \frac{t_2}{2\sqrt{2|t_1|\sqrt{|t_1 t_5|}}}, \quad y = \frac{t_4}{2\sqrt{2|t_5|\sqrt{|t_1 t_5|}}}, \quad z = \frac{t_3}{4\sqrt{|t_1 t_5|}}, \tag{22}$$

$$A = \frac{t_1(2\sigma_5\sigma_7 - \sigma_6^2) + R_3}{8\sigma_5^2 |t_1| \sqrt{|t_1 t_5|}}, \quad B = \frac{t_5^2(2\sigma_{-1}\sigma_{-3} - \sigma_{-2}^2) - R_{-3}}{8\sigma_1^2 |t_5| \sqrt{|t_1 t_5|}},$$

$$C = \frac{-2t_1 t_5 \sigma_{-1}^2 + 8B|t_5|\sqrt{|t_1 t_5|}(2\sigma_{-1}\sigma_{-3} - \sigma_{-2}^2) - t_5^2(\sigma_{-3}^2 + 2\sigma_{-1}\sigma_{-5} - 2\sigma_{-2}\sigma_{-4}) + R_{-4}}{16\sigma_{-1}^2 |t_1 t_5|}. \tag{23}$$

Taking the various combinations of signs in (19), we obtain four systems of quadratic equations, only two of which are essentially different, for example:

$$x^2 - z = A, \quad y^2 - z = B, \quad z^2 - xy = C \tag{24}$$

and

$$x^2 - z = A, \quad y^2 + z = B, \quad z^2 - xy = C. \tag{25}$$

The other two systems are obtained from (24) and (25) by replacing z by $-z$.

We will now show that, if for some A, B, and C, System (25) has real solutions, then System (24) also has such solutions. Considering System (25), one easily finds that, for the existence of real solutions for it, it is necessary and sufficient that the following relationship holds

$$x^2 + y^2 = A + B \geqslant 0 \tag{26}$$

and that the equation

$$\varphi_1(z) = (z^2 - C)^2 - (A + z)(B - z) = 0, \tag{27}$$

obtained by eliminating x and y from System (25), has real roots.

Indeed, if z_1 is a real root of Equation (27), then

$$(z_1 + A)(B - z_1) \geqslant 0. \tag{28}$$

If now the direction of the inequality is the same as in the Condition (26), then one of the quantities x^2 and y^2 is positive (for example, $x^2 > 0$). Then we obtain from (25), $z_1 + A > 0$, and from (28), $-y^2 = (B - z_1) \geq \geq 0$, i.e., the root z_1 indeed provides a real solution to the system. If $A + B = 0$, then by analogous reasoning one easily shows that there is a real solution only if $A = B = C = 0$. Therefore, we shall henceforth assume that (26) is always satisfied. A result of the previous proof is that, given (26), any real root of Equation (27) gives a real solution to (25).

If A and B are interchanged in (25), Equation (27) can be written as

$$\varphi_2(z) = (z^2 - C)^2 - (B + z)(A - z) = 0 \tag{29}$$

and will have roots with the opposite signs of those of Equation (27).

It is obvious from this that the newly obtained system also will have real solutions. Consequently, if (25) has real solutions, then one of the Equations (27) or (29) will have nonnegative roots. Now, let us return to (24). It has real solutions if and only if the equation

$$\varphi(z) = (z^2 - C)^2 - (z + A)(z + B) = 0 \tag{30}$$

has a real root z_2 for which

$$z_2 + A \geqslant 0 \tag{31}$$

and

$$z_2 + B \geqslant 0. \tag{32}$$

If one of these inequalities is satisfied then, by virtue of (30), so is the other. We now prove that if (25) has real solutions, then Equation (30) has nonnegative roots. From (28), (29), and (30), we have

$$\varphi(z) = \varphi_1(z) - 2z(A + z), \tag{33}$$
$$\varphi(z) = \varphi_2(z) - 2z(B + z). \tag{34}$$

From (26), there are two possible cases:

$$A \geqslant 0, \quad B > 0, \tag{35}$$
$$A > 0, \quad B < 0. \tag{36}$$

The cases $A > 0$, $B \geq 0$, and $A < 0$, $B > 0$ are analogous to (35) and (36). If (35) holds and z_0 is a nonnegative root of Equation (27), then from (33) we have

$$\varphi(z_0) = -2z_0(A + z_0) \leqslant 0.$$

Since $\varphi(+\infty) > 0$, Equation (30) in this case has a nonnegative root, $z_2 \geq z_0$. If (36) holds, then Equation (27) has only negative roots, and Equation (29) has only positive roots. If z'_0 is one of the roots of Equation (27), then

$$B + z'_0 \geqslant 0, \tag{37}$$

since, as was proved, if (26) holds then any real root of Equation (27) gives real solutions to (25).

Substituting the value of z'_0 in (34), we get

$$\varphi(z_0') = -2z_0'(B + z_0') \leqslant 0,$$

that is, Equation (30) has a positive root, $z_2 \geq z'_0$. Whence, from (26), follows the existence of real solutions of (24), since this implies that the sum of A or B with the nonnegative root will be positive, for example, $A + z_2 > 0$ and, by virtue of (30), $B + z_2 \geq 0$.

Thus, in order that the solution of the resolving systems, (9) and (10), be real, given (15) and (16), it is necessary and sufficient that (24) have real solutions. Thus, solving the problem of the stability of the trivial solution of (1) leads to the investigation of the quadratic Equations (24). From a detailed consideration of (24) we formulate our final result as a theorem, to be proved below.

Let either (15) or (16) hold. Then, if

$$C + \tfrac{1}{2} > 0, \quad C + \tfrac{1}{4} + AB > 0 \tag{38}$$

and

$$D = 4\left[\frac{(2C+1)^2}{3} + 4C^2 - 4AB\right]^3 - 27\left[\frac{2(2C+1)^3}{27} - \frac{8(2C+1)(C^2-AB)}{3} - (A+B)^2\right] \geqslant 0, \tag{39}$$

then for stability "in the large" of the unperturbed motion corresponding to (1), it is sufficient that at least one of the following conditions does not hold:

$$A + B < 0, \quad AB > C > \tfrac{1}{4}; \tag{40}$$

if

$$D < 0 \tag{41}$$

or

$$D = 0 \tag{42}$$

and the following relationships do not hold

$$C + \tfrac{1}{2} < 0, \quad C + \tfrac{1}{4} + AB = 0, \quad A + B = 0, \tag{43}$$

then it is sufficient for stability that the following inequality holds:

$$A + B \geqslant 0. \tag{44}$$

Thus, if (42) holds then (38) does not. The stability criteria obtained are the most general of all those that might be obtained for the system under consideration by the use of the Lur'e theorem.

Proof. Consider (24). Eliminating x and y we obtain Equation (30), which can be put in the form:

$$z^4 - (2C+1)z^2 - (A+B)z + C^2 - AB = 0. \tag{45}$$

Using known rules [2], we find that if (38) and (39) hold, then Equation (45) has four real roots. If one of the Conditions (41) or (42) of the theorem to be proved holds, then Equation (30) has two real roots. To solve the problem posed, it is necessary to clarify whether or not (31) and (32) hold for at least one of the real roots of Equation (45) for each A and B.

We make the following preliminary remarks:

1. The region of stability in A, B, C parameter space must be symmetric with respect to the plane, $A = B$, since these parameters enter symmetrically into (24).

2. If the signs of A and B are simultaneously changed, then the signs of all the roots of Equation (45) are reversed. Thus, if at point (A, B, C) to the real root, z_1 of Equation (45) there corresponded a real solution to System (24), then at point $(-A, -B, -C)$, a real solution would not correspond to the root z_1. In particular, bearing in mind the first remark above, we must conclude that if a real solution of System (24) corresponds to each real root, z_i, of Equation (45) at the point (A, B, C) then there will be no real solutions at the point symmetric to this with respect to the plane $A + B = 0$.

3. It follows from (30) that, for any real root, z_1, of Equation (45), excluding the cases $z = \pm\sqrt{C}$, we have

$$(z_1 + A)(z_1 + B) > 0. \tag{46}$$

4. Equation (45) has roots $z = \pm\sqrt{C}$, if and only if the corresponding point (A,B,C) lies on one of the parabolic cylinders

$$A^2 = C, \quad B^2 = C. \tag{47}$$

If (47) holds, there are real roots, $z = -A$ $(z = -B)$, which follows from (30).

5. If we change variables in (45), setting $z = \xi - A$, then Equation (45) with respect to ξ will have the form:•

$$\xi^4 = 4A\xi^3 + 2\xi^2\left(3A^2 - C - \frac{1}{2}\right) - \xi(4A^3 - 4AC + B - A) + (A^2 - C)^2 = 0 \tag{47a}$$

Since the constant term of this equation is positive for $A^2 \neq C$, then for all real roots of Equation (45) we have, for $A^2 \neq C$,

$$\prod_{l=1}^{k} = \prod_{l=1}^{k} (z_l + A) > 0 \tag{48}$$

and for $B^2 \neq C$, by virtue of (46),

$$\prod_{l=1}^{k} (z_l + B) > 0. \tag{49}$$

where $\underline{k}$ is the number of real roots of Equation (45). By a simple transformation, Equation (45) may be put in the form

$$\left(z^2 - \frac{2C+1}{2}\right)^2 = (A+B)z + C + \frac{1}{4} + AB. \tag{50}$$

For any real root, z_1, of Equation (45), there holds the inequality

$$(A + B)z_1 + C + \frac{1}{4} + AB \geqslant 0. \tag{51}$$

If $A + B > 0$, then it is not difficult to obtain from (51):

$$2z_1 + A + B \geqslant \frac{A^2 + B^2 - 2\left(C + \frac{1}{4}\right)}{A + B}.$$

From this last inequality, and taking (46) into account, one may infer that for

• The problem set here might be solved by taking as a necessary and sufficient condition the existence of at least one positive root of Equation (47a). However, using the corresponding criteria [3] leads to a really complex relationship which is unsuited for practical use.

$$A+B>0, \quad A^2+B^2-2\left(C+\frac{1}{4}\right)\geqslant 0 \tag{52}$$

to any real root of Equation (45) corresponds a real solution of System (24). But then we have, from our second remark, that for

$$A+B<0, \quad A^2+B^2-2\left(C+\frac{1}{4}\right)\geqslant 0 \tag{53}$$

(24) may not have real solutions. Conditions (52) and (53) determine, in parameter space, a region outside the paraboloid of revolution

$$\frac{A^2+B^2}{2}=C+\frac{1}{4}, \tag{54}$$

also including its surface.

It remains to consider the domain inside the paraboloid, characterized by the inequality

$$\frac{A^2+B^2}{2}<C+\frac{1}{4}. \tag{55}$$

In the sequel we shall always assume that Inequality (55) is satisfied, from which it follows that

$$C+\frac{1}{4}>0 \tag{56}$$

and, moreover,

$$C+\frac{1}{4}+AB>0. \tag{57}$$

Comparing the last two relationships with (38), we find that, at any point of Region (55), Equation (45) has at least one pair of real roots.

In what follows we shall use a geometric presentation. We introduce the notation

$$y_1=\left(z_2-\frac{2C+1}{2}\right)^2, \tag{58}$$

$$y_2=(A+B)z+C+\frac{1}{4}+AB. \tag{59}$$

The investigation of Equation (45), together with (50) leads to the study of the intersections of the fourth order curve (58) with the straight line (59). Taking (56) into account, it is not difficult to find that the curve y_1 is tangent to the z axis for $z=z_0=\pm\sqrt{C+\frac{1}{2}}$. For $z=0$, y_1 attains a maximum, $y_{max}=OA=\frac{1}{2}(2C+1)^2$.

The graph of y_1 has the form shown in Fig. 1. Between the points A and z_0, and also between the points A and $-z_0$, the curve has points of inflection for $z_{inf}=\pm\frac{z_0}{\sqrt{3}}$.

When A, B, and C are in the Region (55), line y_2 cuts the y axis in the point M, above the origin, as required by (57). Here, the distance from point M to point A on the y axis equals

$$AM=\frac{(2C+1)^2}{4}-C-\frac{1}{4}-AC=C^2-AB, \tag{60}$$

that is, it equals the constant term in Equation (45). Consequently, if $C^2-AB>0$, then the point M lies between A and B (line I– I on Fig. 1). Otherwise, point M_1 lies above point A (line II – II on Fig. 1).

It is also clear from (59) that the quantity A + B determines the angle between y_2 and the z axis. For $A+B<0$, line y_2 is turned clockwise with respect to the z axis, and conversely for $A+B>0$.

Consideration of Fig. 1 leads to the following conclusions.

1. For $A + B > 0$, Equation (45) has at least one positive root. Hence, in this case, System (24) has real solutions, since the positive root, z_1, added to the positive quantity A (or B) gives $z_1 + A > 0$, but then by (46), $z_1 + B \geq 0$.

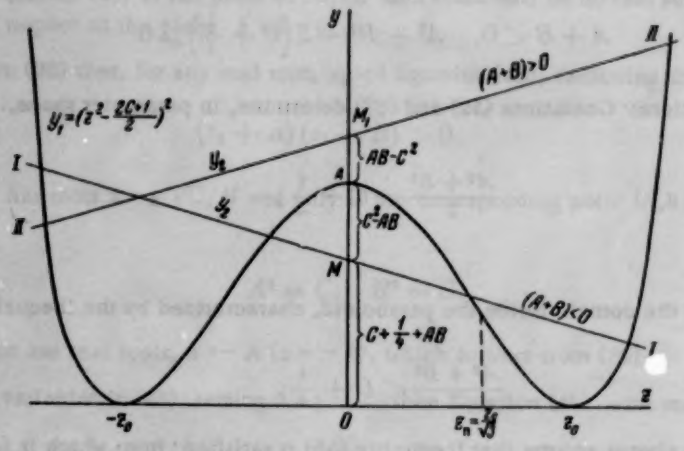

Fig. 1.

2. If Conditions (38) and (39) are fulfilled and, in addition, $AB \leq 0$, System (24) has real solutions, since there is also a positive root in this case, and one of the quantities, A and B, is nonnegative. It thus follows from (53) that for $A + B < 0$, $AB \leq 0$, Equation (45) can have only four real roots inside Region (55).

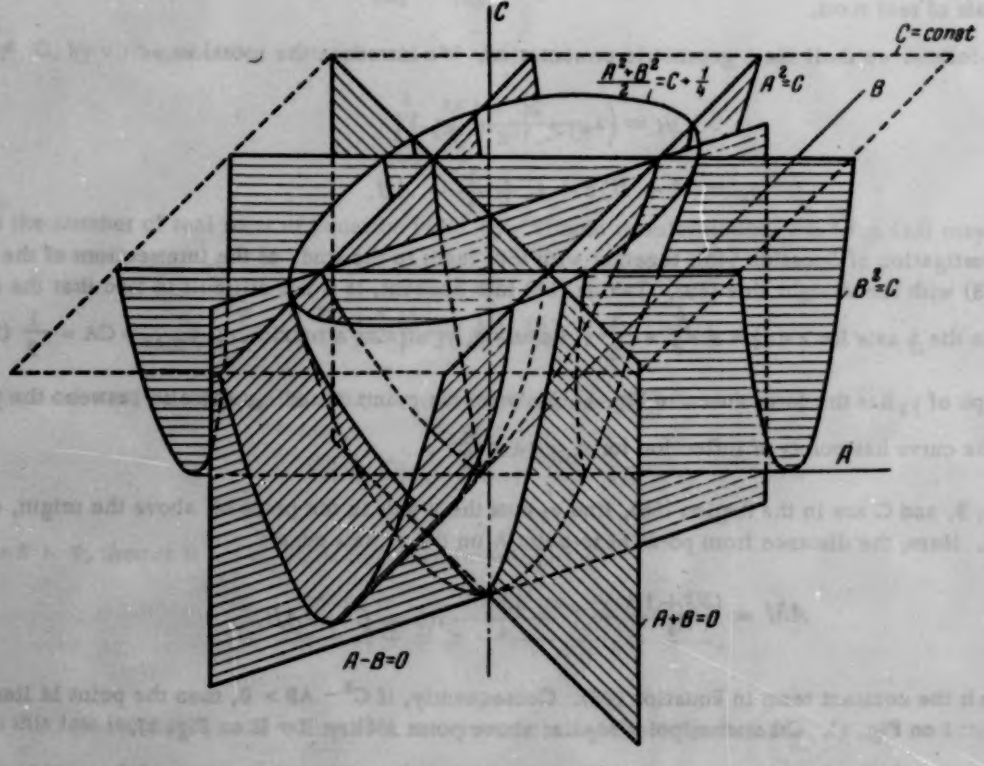

Fig. 2.

3. If $A + B < 0$, $A^2 \neq C$ or $B^2 \neq C$ and Equation (45) has two real roots, then neither of these roots leads to real solutions of System (24). Indeed, it is easily seen from Fig. 1 that at least one of the real roots, z_1, of Equation (45) is negative for $A + B < 0$. Consequently, when added to the negative quantity A (or B), it gives $z_1 + A < 0$, but then, by (48) and (49), for the second root also one will have $z_2 + A < 0$ or $z_2 + B < 0$. Thus, it remains to consider the case when

$$AB > 0, \quad A + B < 0 \tag{61}$$

and the equation has four real roots. We shall consider the section of Region (55) by the plane C = const (Fig. 2). Here the curve y_1 is uniquely determined. Changing A and B within the curve formed by the section, we shall change the attitude of line y_2 and thereby observe the manner in which the roots of Equation (45) change. In view of (61) and the symmetry of the situation, it suffices to consider only the curve lying in the third octant below the line A = B. Suppose, for example, that C lies in the interval $1/4 < C < 1$. Here, the corresponding section (Fig. 3) will be intersected by the surface $AB = C^2$. (It may be shown in general that for $C > \frac{1 + \sqrt{2}}{2}$, Surface (54) does not intersect the surface $AB = C^2$).

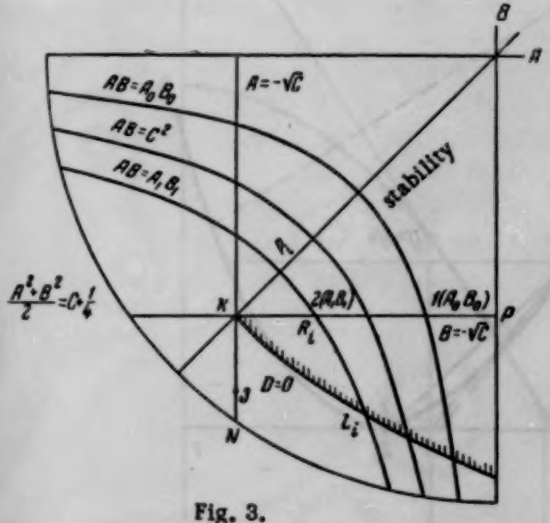

Fig. 3.

On Fig. 3 is also plotted the line

$$A = -\sqrt{C}, \quad B = -\sqrt{C}, \tag{62}$$

obtained by the intersection of the plane C = const with the cylinder (47). Figure 4 gives the curve y_1 corresponding to the given C. Here we limit ourselves to the right branch of the curve, inasmuch as we are only interested in positive roots. For given $A = A_0$, $B = B_0$ (C is fixed), the line y_2 also takes a determinate attitude. It follows from Equation (59) that to a movement of the line

$$A + B = A_0 + B_0 \tag{63}$$

in the plane C = const corresponds a displacement of the line y_2 parallel to itself along the y axis, but to the motion of the hyperbola

$$AB = A_0 B_0 \tag{64}$$

corresponds a rotation of the line y_2 around the point M ($z = 0$, $y = C + \frac{1}{4} + A_0 B_0$). If the point (A, B) is moved along one of the lines (62) in the direction of increasing $|A|$ or $|B|$, then the line y_2 turns in a clockwise sense around the point T ($y = \frac{1}{4}$, $z = \sqrt{C}$), lying on the curve y_1. This follows from the fact that, if one substitutes, for example, $A = -\sqrt{C}$ in (59), one gets $y_2 = (z - \sqrt{C})(B - \sqrt{C}) + \frac{1}{4}$, i.e., an actual turning about the point T. That T lies on the curve y_1 is immediately verified. Thus, if $C = \frac{1}{4}$, the point T coincides with the point of inflection z_{inf} of curve y_1, but if $C > \frac{1}{4}$, then T lies to the right of z_{inf}, that is, in the region where $y_1 < 0$.

In order to establish whether or not a given positive root of Equation (45) provides real solutions of the system, it is obviously sufficient to establish whether z_1, the given root, lies to the right or to the left of one of the lines

$$z = -A, \quad z = -B \tag{65}$$

We consider the attitude of the line y_2 (Fig. 4, line I – I) corresponding to the point 1 ($A_1 - \sqrt{C}$, C) (Fig. 3). At this point, $AB < C^2$ and, consequently, there are two negative and two positive roots. For the negative roots, z_1 and z_2, $(z_1 + A)(z_2 + A) > 0$.

For the root $z_3 = \sqrt{C} = -B$, we have $z_3 + A > 0$, since inside the sector under consideration, $-A < -B$.

This means that for the second positive root, by virtue of (48) and (49), we have the inequalities

$$z_4 + A > 0, \quad z_4 + B > 0 \quad (z_3 \neq z_4, \text{ since } A \neq B).$$

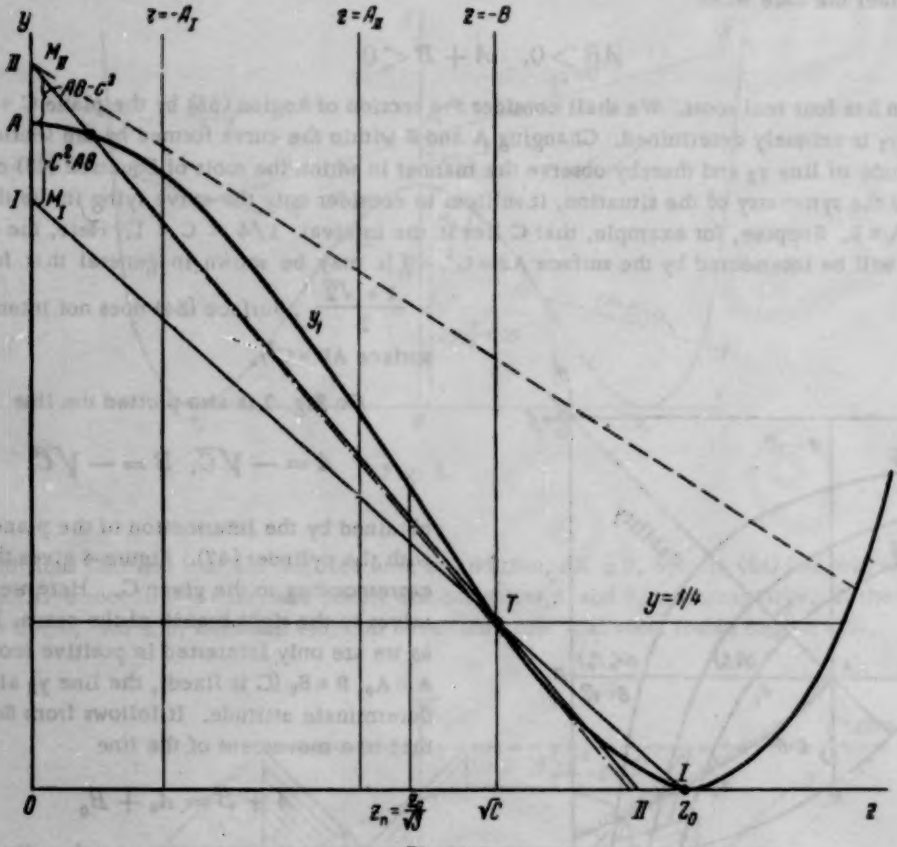

Fig. 4.

Thus, at the point 1, both positive roots lead to real solutions.

Continuously moving from point 1 to point 2 (Fig. 3) along line $B = -\sqrt{C}$, corresponding to a rotation of y_2 about the point T, we find that, at the intersection with the hyperbola $AB = C^2$, the smaller, in absolute value, of the negative roots goes positive, but that the two other positive roots do not change their orientation with respect to lines (65), i.e., as before they give real solutions to the system (Fig. 4, line II – II). Thus, at each point of the segment PK (Fig. 3), including the point K ($A = B = -\sqrt{C}$, C), we have real solutions for the system. Making use of the geometric representation, we may say that if we move from point P_1 (α, α) of the line A = B along the hyperbola $AB = \alpha^2$ (α is some constant), intersecting the segment PK in the point R_1, we shall always have real solutions until we reach a certain point, L_1, coming after point R_1, but inside curve (54). The point L_1 lies on the discriminant surface, D = 0, although at this point the System (24) still has a real solution. Along the remaining portion of the hyperbola leg under consideration there are no real solutions. For proof, consider, for example, point 2 (Fig. 3) and the line corresponding to it, II – II (Fig. 4). If one moves from point 2 along the hyperbola (64) to line A = B, line II – II will rotate counter-clockwise about point M_{II}. During all this, there will be four real roots. In fact, if it is assumed that, in the process of moving, line II – II may occupy the position, shown in dotted lines on Fig. 4, then for the two real roots we would have in this position, we find that $(z_1 + A)(z_2 + A) < 0$, which contradicts (48).

Consequently, the relative distribution of the roots of Equation (45) and lines (65) on the portion of the hyperbola P_1R_1 remains invariant, since it could change only with the intersection of line (62) or by the loss of two real roots. From this follows the existence of real roots in the portion P_1R_1. Moving from point R_1 in the opposite direction along the hyperbola, one will always have four real roots and, consequently, real solutions,

until point L_1 is reached, when line y_2 becomes tangent to curve y_1 (Fig. 3, striped line). At point L_1, obviously, $D = 0$, and it lies within the curve since, as was proved above, for $A + B < 0$, System (24) does not have real solutions external to or on Surface (54). Thus, the required proof. We consider now a certain segment KN on the line $A = -\sqrt{C}$ (Fig. 3). In this case, the root $z_3 = +\sqrt{C} = -A$ does not give real solutions, since $B - A < 0$.

The negative root z_1 and the positive root z_2, which is lesser in absolute magnitude, give $(z_1 + B)(z_2 + B) > 0$ (Fig. 5). For the remaining positive root z_4 we have $z_4 + B < 0$ and $z_2 < z_4 < z_3$ (Fig. 5). Therefore, on the segment KN, the System (24) does not have real solutions (excluding point K).

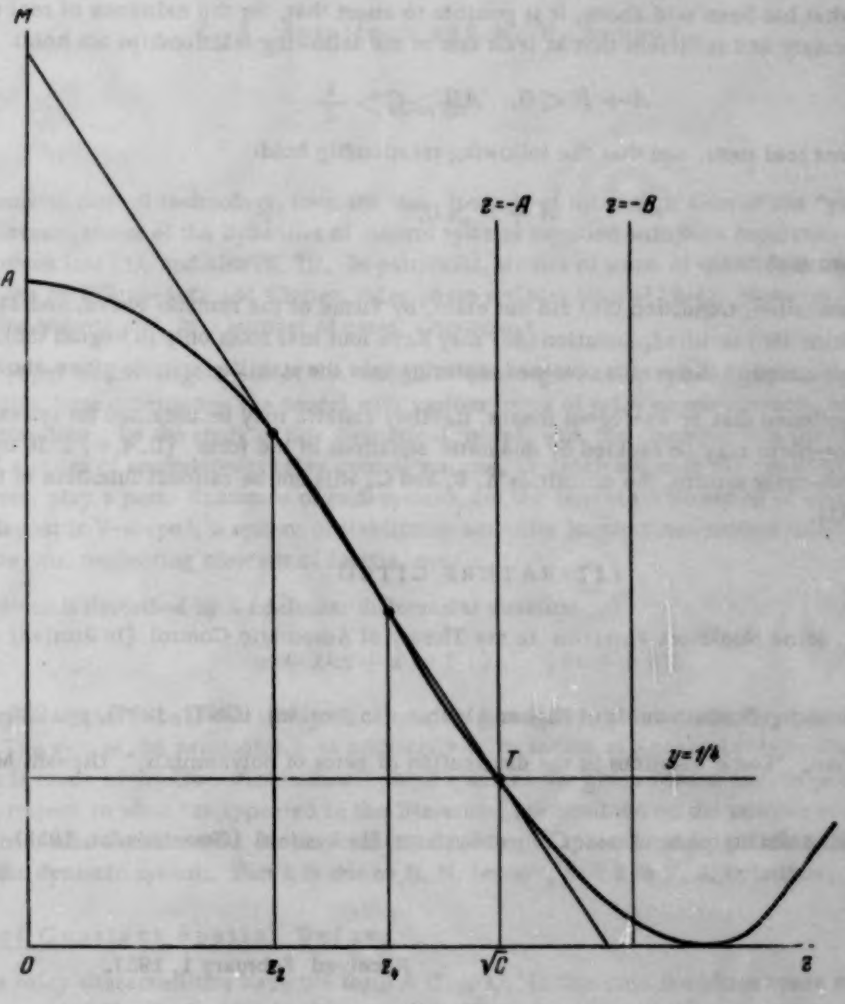

Fig. 5.

Arguing as above, one may show that on any hyperbola (64) which intersects segment KN there are no real solutions. Moreover, it is possible to show that, if there is no solution at the point A_0B_0, then by moving along line (63) to the intersection with line $A = B$, still no solution will exist. Hence, it is not difficult to conclude that, in general, for $AB > C$, real solutions of System (24) cannot be found in the sector under consideration. Consequently, the graph of the domain of stability has the form shown by the striped line in Fig. 3, and is determined by the relationships

$$D = 0, \quad AB \leqslant C. \tag{66}$$

The reasoning given above may be extended to cover all $C > \frac{1}{4}$. For $-\frac{1}{4} \leq C < 0$, and for $0 \leq C < \frac{1}{4}$,

the situation is somewhat different, since in the first case the plane C = const does not intersect the cylinder (47) and, in the second case, the point T ($y = \frac{1}{4}$, $z = \sqrt{C}$) lies on the portion of the curve y_1 for which $y_1'' > 0$, i.e., to the left of the point of inflection.

It may be shown that all points of the strip, $-\frac{1}{4} \leq C \leq \frac{1}{4}$, at which Equation (45) has four real roots, are found in Region (55), and at each of these points System (24) has real solutions. Thus, all possible sign combinations of the quantities A + B and AB have been treated. If A + B = 0, then it is easily shown that a necessary and sufficient condition for the existence of real solutions to the system is the existence of real roots of Equation (45).

Summarizing what has been said above, it is possible to assert that, for the existence of real solutions of System (24), it is necessary and sufficient that at least one of the following relationships not hold:

$$A + B < 0, \quad AB > C > \frac{1}{4}, \tag{67}$$

if Equation (45) has four real roots, and that the following relationship hold:

$$A + B \geqslant 0, \tag{68}$$

if Equation (45) has two real roots.

In the final formulation, Condition (55) did not enter, by virtue of the remarks above, and also because for AB < C and Condition (65) satisfied, Equation (45) may have four real roots only in Region (55). Inequalities (15) and (16) are incorporated in the results obtained, entering into the stability criteria given above.

It should be mentioned that by analogous means, stability criteria may be obtained for systems of arbitrary order [4], only if the problem may be tackled by quadratic equations of the form {II. 4. 6}. In truth, in the general case of an n'th-order system, the quantities A, B, and C, will not be rational functions of the coefficients of the initial System (1).

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ON THE THEORY OF ONE-RELAY SYSTEMS

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(Gor'kii)

In automatic control technology, frequent use is made of relay regulators of the "yes-no" type. Theoretical investigations of the dynamics of control systems supplied with such regulators have been made by a series of authors (see [1], and also [6, 7]). In particular, studies of some of the simplest models of such systems were carried out by Flügge-Lotz and Klotter, using phase surfaces (space) [2-4]. However, the results obtained in [2-4] are inadequate and, in a number of cases, erroneous.*

In this paper we investigate one of the models of one-degree relay systems previously studied by Flügge-Lotz and Klotter, here considering the model with various forms of relay characteristics, using line-to-line point transformations. To the study of this theoretical model, with the corresponding idealizations, is brought the theory of a series of second-order relay control systems, in which not only the coordinates, but also their first derivatives, play a part. Examples of such systems are the lateral stabilization of an aircraft autopilot where wing layout is V-shaped, a system of stabilizing autopilot longitudinal motion with a constant angle of attack for the jets, neglecting moment of inertia, etc.

The system is described by a nonlinear differential equation

$$\ddot{x} + 2h\dot{x} + x = F(\psi), \quad \psi \equiv x + k\dot{x},$$

with the relay characteristics shown in Fig. 1. Here, for generality, h is an arbitrary positive number (in [2-4], $0 < h < 1$). The sign of the parameter k is arbitrary.** By means of Koenig-LeMaire diagrams a detailed investigation is made of the (two-dimensional) phase space of the given system and, in particular, new results, at least with respect to what has appeared in the literature, are obtained on the number of stable limiting cycles. There is also constructed a decomposition of parameter space into regions of qualitatively identical behavior of the dynamic system. Part 1 is due to N. N. Leonov, part 2 to V. S. Boiarinov.

1. Case of Constant Spatial Delay

Let the relay characteristics have the form A (Fig. 1). In this case the phase space of the system takes the form of two partially overlapping half-planes, joined along segments of the lines, $\psi = \delta$ and $\psi = -\delta$ (Fig. 2). If the representative point, describing the motion of the system, falls on these lines, the relay transfers, and the representative point moves over to the other half-plane.

In half-plane I, the system motion is described by the differential equation

$$\dot{x} = y, \quad \dot{y} + 2hy + x = 1. \tag{1}$$

* For example, in works [2-4], for the dynamic system considered below, only one limiting cycle was allowed to exist, although there may be two of them, and sometimes even three, there lacked a decomposition of parameter space into regions of topologically similar phase spaces, etc.

** A particular case of this problem was investigated, for $\delta = 0$, by H. Bilharz [8]. The case when $k = 0$, but the characteristic of the nonlinear element has dead zones and loops, was treated by N. S. Gorskaia [6].

Correspondingly, in half-plane II

$$\dot{x} = y, \quad \dot{y} + 2hy + x = -1. \tag{2}$$

a) Let h < 1. The solutions of the equations of motion, (1) and (2) will have the form:

in half-plane I

$$x = 1 + e^{-ht}(a\cos\omega t + b\sin\omega t); \tag{3}$$

in half-plane II

$$x = -1 + e^{-ht}(a_1\cos\omega t + b_1\sin\omega t), \quad \omega = \sqrt{1-h^2}. \tag{4}$$

The phase plot is symmetric with respect to the origin, since Equations (1) and (2) do not change when (x, y) goes over into (− x, − y). All phase trajectories may be divided into ordinary "plane" and for k < 0, "cylindrical" ones. The latter are found on the flattened surface during the generation of the cylinder which is generated by the two plane strips (between $\psi = -\delta$ and $\psi = \delta$), joined in the region of the switching lines, $\psi = -\delta$, $\psi = \delta$. Movement along the cylindrical surface occurs if $\dot{\psi}$ changes sign at the moment the relay transfers. For ordinary ("plane") motion, $\dot{\psi}$ does not change sign at the moment of transfer.

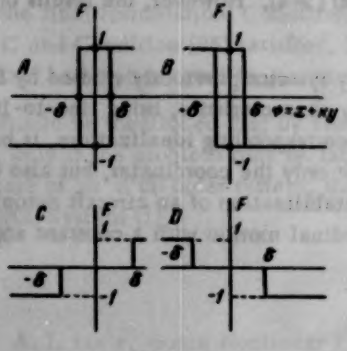

Fig. 1.

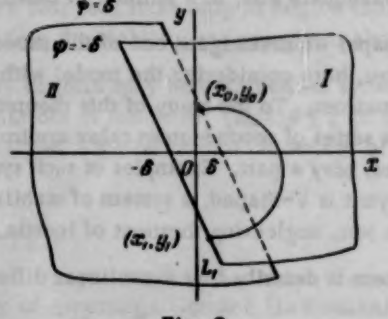

Fig. 2.

The isolated points of the differential Equations (1) are (1,0) and (0,1), and are stable foci. These isolated points have finite zones of attraction. If the initial conditions lie in the zone of attraction of an isolated point, then the system will tend to a stable equilibrium state (x = 1, y = 0, or x = − 1, y = 0), without transfer of the relay. An infinite phase space is unstable, since the equator of the Poincare sphere, in the given case, turns out to be a limiting cycle.

Let the representative point begin moving (t = 0) from the point (x_0, y_0) on the line $\psi = \delta$ and, at time $t = \tau$, land at the point (x_1, y_1) on the line $\psi = -\delta$ (Fig. 2).

Then $a = x_0 - 1$, $b = \frac{1}{\omega}\left[h(x_0 - 1) + \frac{1}{k}(\delta - x_0)\right]$.

Setting $t = \tau$ in (3) and in the derivative of (3), and substituting the values of a and b, we find the expression for the corresponding function in parametric form (with τ the parameter):

$$\begin{aligned} x_0 &= \frac{\omega}{c}\,\frac{(1+\delta)e^{h\tau} - (1-\delta)\cos\omega\tau}{\sin\omega\tau} + \delta + \frac{(1-\delta)(k-h)}{c}, \\ x_1 &= \frac{\omega}{c}\,\frac{-(1-\delta)e^{-h\tau} + (1+\delta)\cos\omega\tau}{\sin\omega\tau} - \delta + \frac{(1+\delta)(k-h)}{c}. \end{aligned} \tag{5}$$

Here

$$c \equiv k - 2h + \frac{1}{k}.$$

The time of transfer τ varies within the interval $\left(\frac{\pi}{\omega}, \tau_1\right)$ for the "plane" case, and within the interval $(0, \tau_2)$ for the "cylindrical" case. τ_1 and τ_2 are determined by the condition that the phase trajectory and the line $\psi = -\delta$ be tangent at the point $(\bar{x}_1, \bar{y}_1)$:

$$\left(\frac{dy}{dx}\right)_{(\bar{x}_1, \bar{y}_1)} = -\frac{1}{k}, \tag{6}$$

from which

$$\bar{x}_1 = -\delta + \frac{k(1+\delta)}{c}.$$

The equations for $\tau_{1,2}$ are of the form:

$$\omega(1-\delta)e^{-h\tau_i} + (1+\delta)(h\sin\omega\tau_i - \omega\cos\omega\tau_i) = 0 \quad (i = 1, 2), \tag{7}$$
$$\pi < \omega\tau_1 < 2\pi, \quad 0 < \omega\tau_2 < \pi.$$

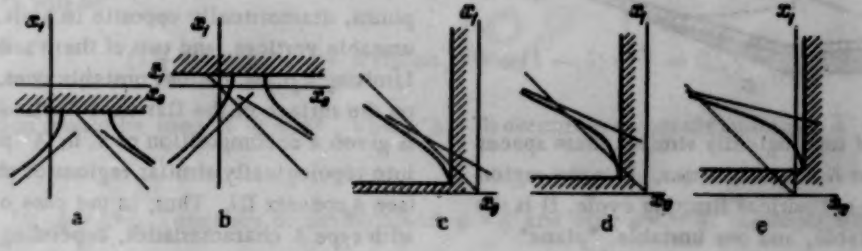

Fig. 3.

By virtue of the phase plot symmetry with respect to the origin, each time a transformation T^+ brings a point into the half-line L_2 (for which $\psi = -\delta$), it is possible to consider instead the point symmetric to it, lying on the half-line L_0 (for which $\psi = \delta$). Thus the problem leads to the consideration of transformations of the half-line L_0 into itself by applications of the transformation $\overline{T^+}$, where the superscript bar indicates a transition, after the application of transformation T^+, to the symmetric point. Therefore, for the construction of the Koenig-LeMaire diagram, we investigate the curve $x_1 = f(x_0)$, described by Equation (5). For this, we find the expressions for $\frac{dx_1}{dx_0}$ and $\frac{d^2x_1}{dx_0^2}$:

$$\frac{dx_1}{dx_0} = \frac{-\omega(1+\delta) + (1-\delta)(h\sin\omega\tau + \omega\cos\omega\tau)e^{-h\tau}}{\omega(1-\delta) + (1+\delta)(h\sin\omega\tau - \omega\cos\omega\tau)e^{h\tau}}, \tag{8}$$

$$\frac{d^2x_1}{dx_0^2} = \frac{c}{\omega}\,\frac{\sin^2\omega\tau\left[-2h(1-\delta^2)\sin\omega\tau + \omega(1+\delta)^2 e^{h\tau} - \omega(1-\delta)^2 e^{-h\tau}\right]}{\left[\omega(1-\delta) + (1+\delta)(h\sin\omega\tau - \omega\cos\omega\tau)e^{h\tau}\right]^3}. \tag{9}$$

Investigation shows (see Appendix I) that the Koenig-LeMaire diagram will have five states (see Fig. 3). It is clear from Fig. 3 that, for $k > 0$, the following cases may occur:

1) two limiting cycles (Fig. 3, d) (external stability and internal instability);

2) one stable limiting cycle (Fig. 3, e)

3) no limiting cycle (Fig. 3, c).

For $k < 0$ (Fig. 3, a, b), there may be either one unstable limiting cycle, distributed on the flattened cylinder, or no limiting cycles.

If $k > 0$ and the initial values lie in the zone of attraction of the isolated points, or internal to an unstable limiting cycle (if such exist), then the system progresses to a stable equilibrium state at the positions

$x = 1$ or $x = -1$. If the initial values lie in the region of attraction of a stable limiting cycle, then the system goes into self-induced oscillation. If $k < 0$, and there exists an unstable limiting cycle on the "cylinder," then the representative point, leaving the cycle via some rotation about the "cylinder," leaves the zone of attraction of the isolated points.

The half-period of the periodic motion is determined from the equation*

$$(h-k)\sin\omega\tau - \omega\delta\cos\omega\tau = \omega\,\mathrm{sh}\,h\tau + \omega\delta\,\mathrm{ch}\,h\tau. \tag{10}$$

For $\delta \to 0$ and $k < 0$, the "cylindrical" surface shrinks to the straight line $\psi = 0$. The motion on the "cylindrical" surface, in the limit, goes over into a sliding regimen.

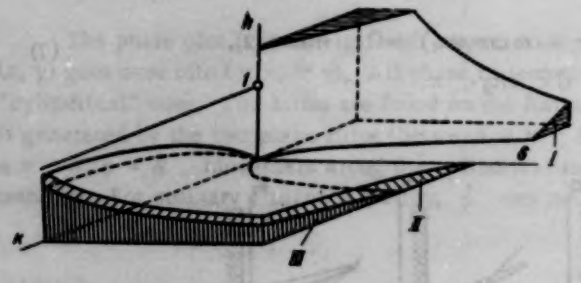

Fig. 4. Regions of topologically similar phase spaces in the case of type A characteristics. I is the region with one unstable cylindrical limiting cycle, II is the region with one stable, and one unstable, "plane" limiting cycle, III is the region with one stable "plane" limiting cycle.

b) Let $h > 1$. The investigation is completely analogous to that given above for the case $h < 1$. The isolated points of the system of differential Equations (1) and (2) are the vertices of stable junctions. The infinite phase space is unstable, since on the equator of the Poincare sphere there are four singular points, diametrically opposite in pairs, two of them unstable vertices, and two of them saddle points. Limiting cycles, besides unstable ones, do not exist on the surface of the flattened "cylinder." In Fig. 4 is given a decomposition of k, h, δ parameter space into topologically similar regions of phase space (see Appendix II). Thus, in the case of a system with type A characteristics, depending on the parameters and on the initial conditions, the system either will tend to the equilibrium states (1,0) or (−1,0), or will develop autooscillation (Fig. 5).

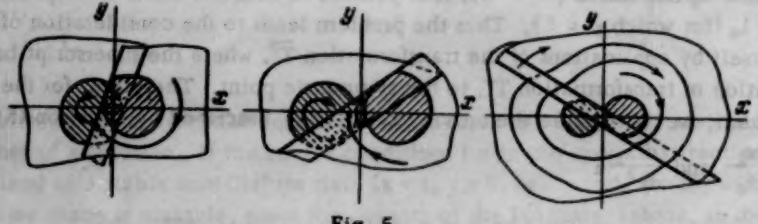

Fig. 5.

Let the relay characteristic have the form B. As above, the phase surface consists of two half-planes, partially over-lapping, and joined along portions of the lines $\psi = \delta$ and $\psi = -\delta$. In half-plane I the system motion is described by the differential equations

$$\dot{x} = y,\ \dot{y} + 2hy + x = -1. \tag{11}$$

Correspondingly in half-plane II

$$\dot{x} = y,\quad \dot{y} + 2hy + x = 1. \tag{12}$$

Letting $h < 1$, we find expressions for the corresponding functions:

$$x_0 = \frac{(1+\delta)\cos\omega\tau - (1-\delta)e^{h\tau}}{c\sin\omega\tau}\omega + \delta + \frac{(1+\delta)(h-k)}{c},$$

$$x_1 = \frac{-(1-\delta)\cos\omega\tau + (1+\delta)e^{-h\tau}}{c\sin\omega\tau}\omega - \delta + \frac{(1-\delta)(h-k)}{c}, \tag{13}$$

* In this paper, 'sh' ≡ 'sinh,' 'ch' ≡ 'cosh' – Publisher.

from which

$$\frac{dx_1}{dx_0} = \frac{\omega(1-\delta)+(1+\delta)(-h\sin\omega\tau-\omega\cos\omega\tau)e^{-h\tau}}{-\omega(1+\delta)+(1-\delta)(-h\sin\omega\tau+\omega\cos\omega\tau)e^{h\tau}}, \tag{14}$$

$$\frac{d^2x_1}{dx_0^2} = \frac{c}{\omega}\,\frac{\sin^3\omega\tau\,[-2h(1-\delta^2)\sin\omega\tau+\omega(1-\delta)^2e^{h\tau}-\omega(1+\delta)^2e^{-h\tau}]}{[-\omega(1+\delta)+(1-\delta)(-h\sin\omega\tau+\omega\cos\omega\tau)e^{h\tau}]^3}. \tag{15}$$

As in the previous case, it is possible to have, not only the ordinary "plane" movement on the phase surface, but also movement along a "cylindrical" surface, for certain parameter values and for $k > 0$. The isolated points of the system of differential equations (11) and (12) are the stable foci type, with coordinates $(1,0)$ and $(-1,0)$. These isolated points, for $\delta < 1$, do not lie in their region of definition.

The time of transfer τ for movement of the representative point along the "cylindrical" surface, is within the interval $(0, \tau_1)$, where $\tau_1 \leq \frac{\pi}{\omega}$, defined by the condition that the phase trajectory be tangent to the lines $\psi = \delta$, $\psi = -\delta$. We distinguish three sub-cases.

1. Tangency to $\psi = \delta$ at the point $(\bar{x}_0, \bar{y}_0)$ (Fig. 6, a). The condition gives $\bar{x}_0 = \delta - k(1+\delta)/c$, and the equation for τ_1' will be

$$(1+\delta)(\omega\cos\omega\tau_1' + h\sin\omega\tau_1') - \omega(1-\delta)e^{h\tau_1'} = 0. \tag{16}$$

This condition should be used if $\delta < \delta^*$, where δ^* is determined from the equation $\delta^* = \frac{1+e^{-\pi h/\omega}}{1-e^{-\pi h/\omega}}$.

2. $\tau_1 = \frac{\pi}{\omega}$ for $\delta = \delta^*$. Tangency occurs both with $\psi = \delta$ and with $\psi = -\delta$, corresponding to the points $(\bar{x}_0, \bar{y}_0)$ and $(\bar{x}_1, \bar{y}_1)$.

3. Tangency with $\psi = -\delta$ at the point $(\bar{x}_1, \bar{y}_1)$ which gives

$$\bar{x}_1 = -\delta - \frac{k(1-\delta)}{c}.$$

The equation for τ_1'' has the form:

$$-\omega(1+\delta)+(1+\delta)(-h\sin\omega\tau_1'' + \omega\cos\omega\tau_1'')e^{h\tau_1''} = 0 \tag{17}$$

and is used in case $\delta > \delta^*$.

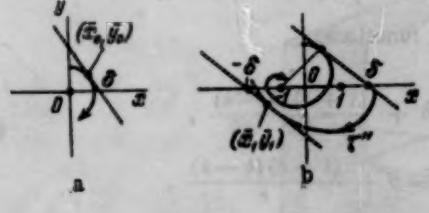

Fig. 6.

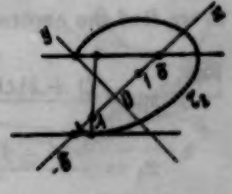

Fig. 7.

The time of transfer τ for "plane" motion changes continuously, with changing x_0, from τ_1' to $\frac{\pi}{\omega}$, if $\delta < \delta^*$, and from $\frac{\pi}{\omega}$ to τ_2, if $\delta > \delta^*$; where τ_2 is determined by the condition of tangency with $\psi = -\delta$ at the point $(\bar{x}_1, \bar{y}_1)$. The equation for τ_2 coincides with Equation (17), but $\tau_2 > \frac{\pi}{\omega}$ (Fig. 7). The time of transfer τ_2 tends to $\frac{\pi}{\omega}$ for motion of the point (x_0, y_0) along the line $\psi = \delta$ out to infinity.

Investigations show (see Appendix III) that the LeMaire diagram may have the form given in Fig. 8. It is clear from Fig. 8 that, for k > 0, there may be the cases: one stable limiting cycle on the "cylinder" (Fig. 8, b, c), one stable limiting cycle and two unstable on the "cylinder" (Fig. 8, d), one unstable limiting cycle on the "cylinder" (Fig. 8, f), one stable "plane" limiting cycle (Fig. 8, a), no limiting cycle at all (Fig. 8, g).

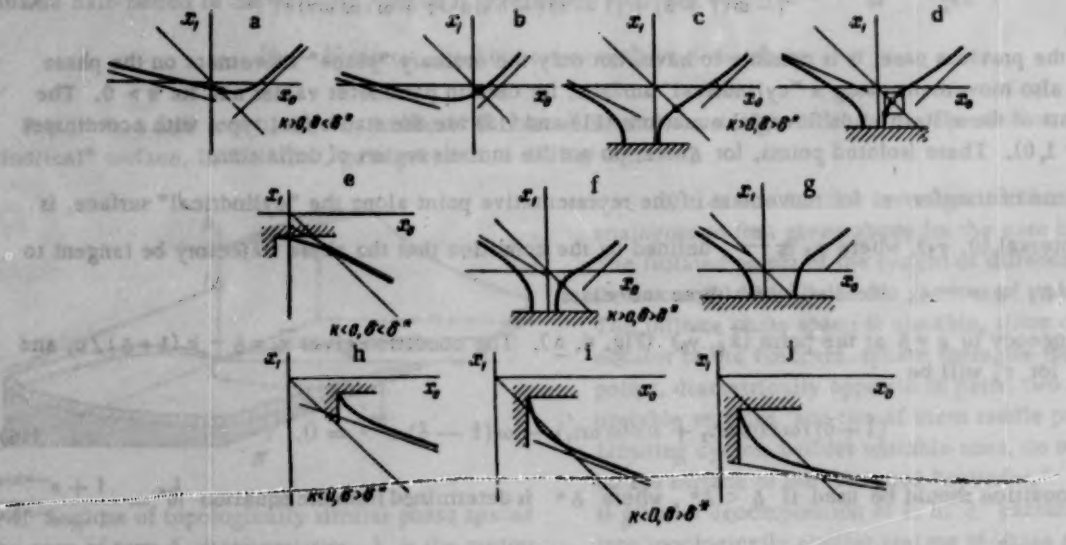

Fig. 8.

For k < 0, there may be the cases: one stable "plane" limiting cycle (Fig. 8, e, j), one stable (external) and one unstable (internal) "plane" limiting cycle (Fig. 8, i), no limiting cycles (Fig. 8, h).*

If the initial conditions lie in the zone of attraction of the stable limiting cycles, the system will develop auto-oscillation. If the initial conditions lie in the zone of attraction of the isolated points, internal to the unstable "plane" limiting cycles or parts of the "cylinder" not included between unstable "cylindrical" limiting cycles, then the system will move to stable equilibrium states at the positions x = 1 or x = − 1. For $\delta \to 0$, k > 0, the "cylindrical" surface shrinks to the line $\psi = 0$. In the limit, motion on it tends to the sliding regimen, leading to the stable equilibrium state (0,0) to which, for $\delta = 0$, the stable "cylindrical" limiting cycle degenerates. With decreasing δ, the limiting cycles shrink, and the frequency of auto-oscillation increases without bound, so that it is possible to consider the equilibrium state (0,0), for $\delta = 0$, as a degenerate limiting cycle, corresponding to auto-oscillation with an infinitely high frequency and an infinitesimally small amplitude.

Assuming h > 1, we find the expressions for the corresponding functions:

$$x_0 = \frac{(1+\delta)\operatorname{ch}\omega_1\tau - (1-\delta)e^{h\tau}}{c\operatorname{sh}\omega_1\tau}\omega_1 + \delta + \frac{(1+\delta)(h-k)}{c}, \qquad (18)$$

$$x_1 = \frac{-(1-\delta)\operatorname{ch}\omega_1\tau + (1+\delta)e^{-h\tau}}{c\operatorname{sh}\omega_1\tau}\omega_1 - \delta + \frac{(1-\delta)(h-k)}{c},$$

where $\omega_1 = \sqrt{h^2-1}$, $c = k - 2h + \frac{1}{k}$, whence

$$\frac{dx_1}{dx_0} = \frac{\omega_1(1-\delta) - (1+\delta)(\omega_1\operatorname{ch}\omega_1\tau + h\operatorname{sh}\omega_1\tau)e^{-h\tau}}{-\omega_1(1+\delta) + (1-\delta)(\omega_1\operatorname{ch}\omega_1\tau - h\operatorname{sh}\omega_1\tau)e^{h\tau}}, \qquad (19)$$

$$\frac{d^2x_1}{dx_0^2} = \frac{c}{\omega_1}\,\frac{\operatorname{sh}^3\omega_1\tau[-2h(1-\delta^2)\operatorname{sh}\omega_1\tau + \omega_1(1-\delta)^2e^{h\tau} - \omega_1(1+\delta)^2e^{-h\tau}]}{[-\omega_1(1+\delta) + (1-\delta)(\omega_1\operatorname{ch}\omega_1\tau - h\operatorname{sh}\omega_1\tau)e^{h\tau}]^3}. \qquad (20)$$

* The cases a, d, f, g, h, and i of Fig. 8 are only possible for $\delta > 1$.

It suffices to consider the case $\delta < 1$ since for $\delta > 1$ limiting cycles do not exist, and the representative point, after no more than one transfer, leaves the stable positions $x = +1$ or $x = -1$.

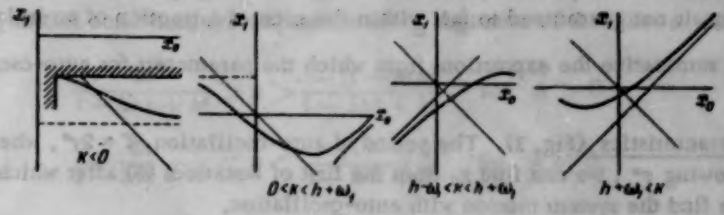

Fig. 9.

The time of transfer for "plane" motion varies within the interval (τ_1, ∞) and, for motion on the "cylinder," only possible for $k > 0$, within the limits from 0 to τ_1. The time τ_1 is determined from the condition that the curve be tangent to the line $\psi = \delta$ at the point $(\bar{x}_0, \bar{y}_0)$. From this condition:

$$\bar{x}_0 = \delta - \frac{k(1+\delta)}{c}.$$

The equation for τ_1 may be written as

$$(1+\delta)(\omega_1 \operatorname{ch} \omega_1 \tau_1 + h \operatorname{sh} \omega_1 \tau_1) - \omega_1 (1-\delta) e^{h\tau_1} = 0.$$

Investigation shows (see Appendix IV) that the LeMaire diagram can have the form shown in Fig. 9..

From these diagrams it is clear that there exists one stable limiting cycle which, for some $\underline{k}$, transfers from the "cylinder" to the plane, or vice versa.

For $\delta < 1$, any initial values lead to a stable limiting cycle and, for $\delta > 1$, to stable equilibrium positions (vertices) with coordinates $(-1, 0)$ or $(+1, 0)$.

For $\delta \to 0$, the "cylinder" generates a sliding regimen, and the stable "cylindrical" limiting cycles tend to the stable isolated point (0,0).

Figure 10 gives the decomposition of parameter space into regions of topologically similar phase spaces (see Appendix V).

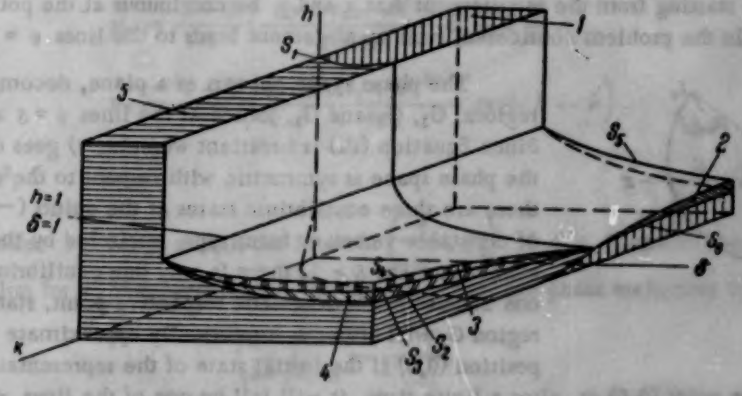

Fig. 10. Regions of topologically similar phase spaces for type B characteristics (Fig. 1). 1) The region with one stable "plane" limiting cycle; 2) the region with one stable and one unstable " "plane" limiting cycle; 3) the region with one unstable "cylindrical" cycle; 4) the region with one stable and two unstable "cylindrical" limiting cycles; and 5) the region with one stable "cylindrical" limiting cycle (equations for the surfaces $S_1, \ldots,$ S_6 are given in Appendix V).

From all that has been said above, it follows that if a relay has type B characteristics (Fig. 1), the representative point, for $\delta < 1$, inevitably tends to a stable limiting cycle. In all other cases (type A characteristics, and type B for $\delta > 1$), despite the presence of stable limiting cycles, the representative point, with corresponding excitations, is not guaranteed to fall within the zone of attraction of an isolated point.

In conclusion, we summarize the expressions from which the parameters for auto-oscillation are determined.

Type A Relay Characteristics (Fig. 1). The period of auto-oscillation, $T = 2\tau^*$, where τ^* is determined from Equation (10). Knowing τ^*, we can find x_0 from the first of Equations (5) after which, determining a and b from Equation (3), we find the system motion with auto-oscillation.

Type B Relay Characteristics. The period of auto-oscillation is $T = 2\tau^*$, where the half-period τ^* is determined from the equation

$$(h-k)\sin\omega\tau + \omega\delta\cos\omega\tau = \omega\,\mathrm{sh}\,h\tau - \omega\delta\,\mathrm{ch}\,h\tau.$$

Knowing τ^*, we find x_0 from the first of Equations (13) after which, determining a and b ($a = x_0 + 1$, $b = \left[h(x_0+1) + \frac{1}{k}(\delta - x_0)\right] / \omega$), we find the system motion with auto-oscillation:

$$x = -1 + e^{-ht}(a\cos\omega t + b\sin\omega t).$$

2. Relay Characteristics With Zones of Instability

Let the characteristic of the nonlinear element have the form of C (Fig. 1). The problem is described by a system of differential equations of the form

$$\dot{x} = y;$$
$$\dot{y} + 2hy + x = F(\psi),$$
$$\psi \equiv x + ky,\quad F(\psi) = \begin{cases} 1 & \text{for } \psi > \delta, \\ 0 & \text{for } |\psi| < \delta, \\ -1 & \text{for } \psi < -\delta. \end{cases} \qquad (21)$$

As above, the solution is found by considering certain linear differential equations with successive "aligned" stationary integrations, starting from the requirement that x and y be continuous at the points of discontinuity of the right side of (21). In the problem considered, such an alignment leads to the lines $\psi = \pm\delta$.

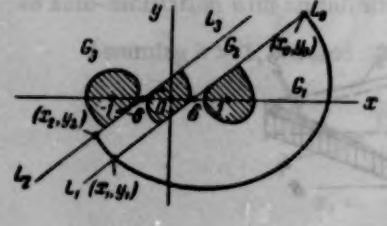

Fig. 11.

The phase space appears as a plane, decomposed into three regions, G_1, G_2 and G_3, joined at the lines $\psi = \delta$ and $\psi = -\delta$ (Fig. 11). Since Equation (21) is invariant when (x, y) goes over into $(-x, -y)$, the phase space is symmetric with respect to the origin. For $\delta < 1$, there are three equilibrium states at the points $(-1,0)$, $(0,0)$, and $(1,0)$, of the stable vertex or focus type, encircled by their respective zones of attraction. For $\delta > 1$, there is only one equilibrium state, $(0,0)$, with one zone of attraction. The respective point, starting its motion in region G_2 will either asymptotically approximate to the equilibrium position $(0,0)$ if the initial state of the representative point lies in the zone of attraction of the point $(0,0)$ or, after a finite time, it will fall on one of the lines $\psi = \pm\delta$ and transferred by it, will move along the trajectory adjoining this line on the other side, following the corresponding other differential equation. The phase trajectory of region G_1 is tangent to the line $\psi = \delta$ in the point with the ordinate

$$y = -\frac{(1-\delta)k}{1 - 2hk + k^2}.$$

The phase trajectory of G_2 is tangent to this line at the point with the ordinate

$$y=\frac{\delta k}{1-2hk+k^2}.$$

Therefore, the aligning initial conditions are impossible along segments of the line $\psi=\delta$ for values of y in the intervals

$$\frac{\delta k}{1-2hk+k^2}>y>\frac{-(1-\delta)k}{1-2hk+k^2} \quad \text{for} \quad k>0,$$

$$\frac{\delta k}{1-2hk+k^2}<y<\frac{-(1-\delta)k}{1-2hk+k^2} \quad \text{for} \quad k<0,$$

since, on these segments of the line $\psi=\delta$, the trajectories adjoining the line are directed in different directions (similarly for the line $\psi=-\delta$). In these cases there occurs a sliding motion of the representative point along the lines satisfying the equation $\dot{x}=\frac{1}{k}(-x\pm\delta)$. The sliding regimen is stable for $k<0$ and unstable for $k>0$. If the sliding regimen is stable, the representative point moves along the lines $\psi=\pm\delta$ to the boundary of the zone of attraction, then enters the zone of attraction. If the sliding regimen is unstable then, with an arbitrarily small excitation, the representative point leaves the lines $\psi=\pm\delta$.

Let $h<1$. Then the solution of differential equation System (21) has the form:

$$\begin{aligned}
x&=1+e^{-ht}(a_1\cos\omega t+b_1\sin\omega t) && \text{for} \quad \psi>\delta,\\
x&=e^{-ht}(a_2\cos\omega t+b_2\sin\omega t) && \text{for} \quad |\psi|<\delta,\\
x&=-1+e^{-ht}(a_3\cos\omega t+b_3\sin\omega t) && \text{for} \quad \psi<-\delta,\\
&\omega=\sqrt{1-h^2}.
\end{aligned} \tag{22}$$

Initially, at time $t=0$, let the representative point be at point (x_0, y_0) of the half-line L_0 and after time τ_1 let it fall on point (x_1, y_1) of half-line L_1 via trajectories in region G_1 (transformation S^+). Then

$$a_1=\delta-1-ky_0, \quad b_1=\frac{y_0+h\delta-khy_0-h}{\omega}. \tag{22'}$$

The corresponding function for the transformation S^+ (in parametric form) is written as follows:

$$\begin{aligned}
y_0&=\frac{1-\delta}{c_1}\left(\frac{\cos\omega\tau_1-e^{h\tau_1}}{\sin\omega\tau_1}\,\omega+h-k\right),\\
y_1&=\frac{1-\delta}{c_1}\left(\frac{-\cos\omega\tau_1+e^{-h\tau_1}}{\sin\omega\tau_1}\,\omega+h-k\right),
\end{aligned} \tag{23}$$

where $c_1=1-2hk+k^2$.

The time of transfer τ_1 varies within the interval $\left(\frac{\pi}{\omega}, \bar{\tau}_1\right)$ if $\delta<1$, and in the interval $\left(0, \frac{\pi}{\omega}\right)$ if $\delta>1$. The equation for τ_1 is determined by the condition that the phase trajectory touch the line $\psi=\delta$:

$$\left(\frac{dy}{dx}\right)_{\bar{x}_1,\,\bar{y}_1}=-\frac{1}{k},$$

whence

$$\bar{y}_1=-\frac{(1-\delta)k}{c_1} \quad \text{and} \quad h\sin\omega\bar{\tau}_1-\omega\cos\omega\bar{\tau}_1+\omega e^{-h\bar{\tau}_1}=0. \tag{24}$$

In investigating the corresponding function, $y_1(y_0)$, we find

$$\frac{dy_1}{dy_0}=\frac{-\omega+(h\sin\omega\tau_1+\omega\cos\omega\tau_1)e^{-h\tau_1}}{\omega+(h\sin\omega\tau_1-\omega\cos\omega\tau_1)e^{h\tau_1}}\equiv\frac{f_2}{f_1}, \tag{25}$$

$$\frac{d^2y_1}{dy_0^2} = \frac{c_1 \sin^3 \omega\tau_1}{\omega(1-\delta)} \frac{2(h \sin \omega\tau_1 - \omega \operatorname{sh} h\tau_1)}{[\omega + (h \sin \omega\tau_1 - \omega \cos \omega\tau_1)e^{h\tau_1}]^3} \equiv \frac{c_1 \sin^3 \omega\tau_1}{\omega(1-\delta)} \frac{f_3}{f_2^3} \tag{26}$$

In the interval $(0, \bar{\tau}_1)$, the following inequalities hold:

$$f_1 > 0,\ f_2 < 0,\ f_3 < 0 \text{ and } \frac{dy_1}{dy_0} < 0, \quad \frac{d^2y_1}{dy_0^2} > 0.$$

The curve $y_1(y_0)$ has the asymptote

$$y_1 = -e^{-\frac{\pi h}{\omega}} y_0 + \frac{1-\delta}{c_1}(2h-k)\left(1+e^{-\frac{\pi h}{\omega}}\right).$$

Now let the representative point leave the point (x_1, y_1) of the half-line L_1 at time $t = 0$ and, at time $t = \tau_2$, fall on the point (x_2, y_2) of the half-line L_2 via trajectories in region G_2 (transformation E^+). With this, $a_2 = \delta - ky$, $b_2 = \frac{y_1 + \delta h - khy_1}{\omega}$. The corresponding function has the form:

$$y_1 = +\frac{\delta}{c_1}\left(\frac{-\cos \omega\tau_2 - e^{h\tau_2}}{\sin \omega\tau_2}\omega - h + k\right),$$

$$y_2 = +\frac{\delta}{c_1}\left(\frac{-\cos \omega\tau_2 - e^{-h\tau_2}}{\sin \omega\tau_2}\omega + h - k\right). \tag{27}$$

The time of transfer τ_2 varies in the interval $(0, \bar{\tau}_2)$ where $\bar{\tau}_2 < \frac{\pi}{\omega}$. Since $\bar{y}_2 = -\frac{\delta k}{c_1}$, the equation for $\bar{\tau}_2$ has the form:

$$-h \sin \omega\bar{\tau}_2 + \omega \cos \omega\bar{\tau}_2 + \omega e^{-h\bar{\tau}_2} = 0. \tag{28}$$

It follows from Equation (27) that

$$\frac{dy_2}{dy_1} = \frac{\omega + (h \sin \omega\tau_2 + \omega \cos \omega\tau_2)e^{-h\tau_2}}{\omega + (-h \sin \omega\tau_2 + \omega \cos \omega\tau_2)e^{h\tau_2}} \equiv \frac{f_4}{f_5}, \tag{29}$$

$$\frac{d^2y_2}{dy_1^2} = \frac{c_1 \sin^3 \omega\tau_2}{\omega\delta} \frac{2(h \sin \omega\tau_2 + \omega \operatorname{sh} h\tau_2)}{[\omega + (-h \sin \omega\tau_2 + \omega \cos \omega\tau_2)e^{h\tau_2}]^3} \equiv \frac{c_1 \sin^3 \omega\tau_2}{\omega\delta} \frac{f_6}{f_5^3}. \tag{30}$$

In the interval $(0, \bar{\tau}_2)$, the following inequalities are satisfied: $f_4 > 0$, $f_5 > 0$, and $f_6 > 0$. Consequently, $\frac{dy_2}{dy_1} > 0$ and $\frac{d^2y_2}{dy_1^2} > 0$. The curve $y_2(y_1)$ has the asymptote $y_2 = y_1 + \frac{2\delta}{c_1}(2h - k)$.

Further, let a transformation S^- take points of the half-line L_2 into points of the half-line L_3 ($\psi = -\delta$) along trajectories in region G_3, and a transformation E^- transfer points of the half-line L_3 into points of the half-line L_0 ($\psi = \delta$) along trajectories in the region G_2. We assume that the points (x_0, y_0) (x_1, y_1), (x_2, y_2), . . . , lie outside the segments of sliding motion. The transformation $T = S^+E^+S^-E^-$ takes the half-line L_0 into itself.

By virtue of the symmetry of the phase plot with respect to the origin, it is always possible, when a point is brought by the transformation $T = S^+E^+$ to a point on the half-line L_2, to consider instead the point symmetric to it, lying on L_0. Therefore, the problem leads to the consideration of transformations of the half-line L_0 into itself via applications of transformations, $T = \overline{S^+E^+}$, where the superscript bar indicates that there is a transfer to the symmetric point after application of the transformation S^+E^+.

If the transformation T is not applicable to certain points of the half-line L_0, this means that the corresponding representative points lie either in a zone of attraction or on lines of the sliding regimen.

Figure 12 gives some of the LeMaire diagrams constructed in accord with Equations (23) and (27). It is clear from Fig. 12 that, for $k < 0$, limiting cycles are not possible. For $k > 0$ there may occur either stability

in the large, or two limiting cycles (stable inside, unstable outside) or one stable limiting cycle. If the initial motion lies within the region of attraction of a stable limiting cycle, the system will fall into auto-oscillation.

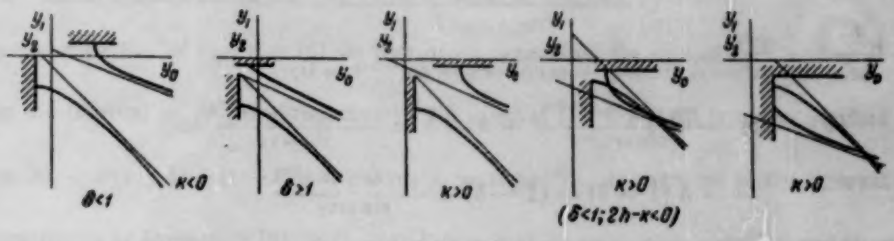

Fig. 12.

Its period will be $T = 2(\tau_1 + \tau_2)$ where τ_1 and τ_2 are the times of transfer corresponding to the fixed points of the transformation. If the initial values lie in the zone of attraction of an isolated point or exterior to an unstable limiting cycle (if such exist), the system goes to a stable equilibrium state at the position (1,0), or at (− 1,0) or, finally, at (0,0).

We now find a decomposition of k, h, δ parameter space into regions of qualitatively identical dynamic system behavior. In accord with Fig. 12, the condition that there appear a semi-stable cycle (the condition of mutually tangent curves) determines the boundary dividing the region in which two limiting cycles exist from the region with no limiting cycles. These conditions may be written as

$$y_0(\tau_1) = -y_2(\tau_2),\ y_1(\tau_1) = y_1(\tau_2),\ \frac{dy_1}{dy_0}(\tau_1) = -\frac{dy_1}{dy_2}(\tau_2)$$

or, in expanded form,

$$(1-\delta)\,\omega\,\frac{\cos\omega\tau_1 - e^{h\tau_1}}{\sin\omega\tau_1} + h - k = \frac{\cos\omega\tau_2 + e^{-h\tau_2}}{\sin\omega\tau_2}\,\delta\omega, \tag{31}$$

$$(1-\delta)\,\omega\,\frac{-\cos\omega\tau_1 + e^{-h\tau_1}}{\sin\omega\tau_1} + h - k = \frac{-\cos\omega\tau_2 - e^{h\tau_2}}{\sin\omega\tau_2}\,\delta\omega, \tag{32}$$

$$\frac{-\omega + (h\sin\omega\tau_1 + \omega\cos\omega\tau_1)\,e^{-h\tau_1}}{\omega + (h\sin\omega\tau_1 - \omega\cos\omega\tau_1)\,e^{h\tau_1}} = -\frac{\omega + (-h\sin\omega\tau_2 + \omega\cos\omega\tau_2)\,e^{h\tau_2}}{\omega + (h\sin\omega\tau_2 + \omega\cos\omega\tau_2)\,e^{-h\tau_2}}. \tag{33}$$

Equations (31) - (33) are the parametric equations for the limits sought (with τ_1 and τ_2 as parameters).

From the condition for the disappearance of unstable cycles (one of the points of intersection of the curves lies on the boundary of the zone of attraction) we determine the boundary dividing the region in which two limiting cycles exist from the region in which one limiting cycle exists. These conditions are written as:

for $\delta < 1$

$$\tau_1 = \bar{\tau}_1,\ y_2(\tau_2) = -y_0(\bar{\tau}_1),\ y_1(\tau_2) = \bar{y}_1(\bar{\tau}_1);$$

for $\delta > 1$

$$\tau_2 = \bar{\tau}_2,\ y_0(\tau_1) = -\bar{y}_2(\bar{\tau}_2),\ y_1(\tau_1) = y_1(\bar{\tau}_2);$$

or, in expanded form:

for $\delta < 1$

$$h\sin\omega\tau_1 - \omega\cos\omega\tau_1 + \omega e^{-h\tau_1} = 0,$$

$$(1-\delta)\,\omega\,\frac{\cos\omega\tau_1 - e^{h\tau_1}}{\sin\omega\tau_1} + h - k = \frac{\cos\omega\tau_2 + e^{-h\tau_2}}{\sin\omega\tau_2}\,\delta\omega, \tag{34}$$

$$k = \delta h + \frac{\cos \omega\tau_2 + e^{h\tau_2}}{\sin \omega\tau_2}\,\omega\delta;$$

for $\delta > 1$

$$\begin{gathered} -h \sin \omega\tau_2 + \omega \cos \omega\tau_2 + \omega e^{-h\tau_2} = 0, \\ (1-\delta)\,\omega \frac{-\cos \omega\tau_1 + e^{-h\tau_1}}{\sin \omega\tau_1} + h - k = \frac{-\cos \omega\tau_2 - e^{h\tau_2}}{\sin \omega\tau_2}\,\delta\omega, \\ k = h(1-\delta) + (1-\delta)\,\omega \frac{\cos \omega\tau_1 - e^{h\tau_1}}{\sin \omega\tau_1}. \end{gathered} \tag{35}$$

In the expressions given above it is necessary, for $h > 1$, to replace ω by $i\omega_1$, where $\omega_1 = \sqrt{h-1}$. The decomposition of parameter space for the case under consideration is given in Fig. 13 (see Appendix VI).

If the relay has type D characteristics (Fig. 1), then the problem is described by the following system of differential equations:

$$\begin{gathered} \dot{x} = y, \\ \dot{y} + 2hy + x = F(\psi), \\ \psi = x + ky, \quad F(\psi) = \begin{cases} -1 \text{ for } \psi > \delta, \\ 0 \text{ for } |\psi| < \delta, \\ 1 \text{ for } \psi < -\delta. \end{cases} \end{gathered} \tag{36}$$

The investigation of this problem also leads to point transformation of a line into itself. By virtue of the symmetry of phase space it is possible, here as previously, to limit oneself to half the point transformations of a half-line L_0 into the half-line L_2 (transformations $T_1 = S_1^+ E_1^+$). If we let $b = 1 - \delta$, and $b_1 = -(1+\delta)$, it is then easy to show that the transformation $T_1 = S_1^+ E_1^+$ is obtained from the transformation $T = S^+E^+$, in the case of type C characteristics, by replacing b by b_1 (for $b < 0$) in the corresponding expressions.

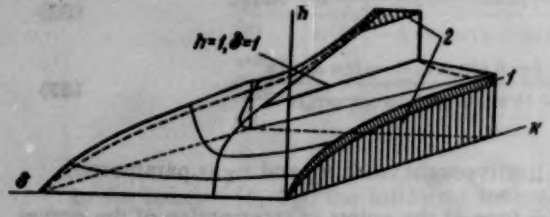

Fig. 13. Regions of topologically similar phase spaces in the case of type C characteristics (Fig. 1). 1) The domain with one stable and one unstable limiting cycle; 2) the region with one stable limiting cycle.

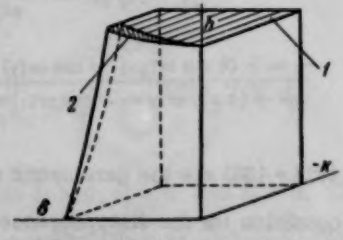

Fig. 14. Regions of topologically similar phase spaces in the case of type D characteristics (Fig. 1). 1) The region with one stable limiting cycle; 2) the region with one stable and one unstable limiting cycle.

Independently, of the magnitude of δ, there exists only one equilibrium state, (0,0), with its own zone of attraction, since the isolated points, (−1,0) and (1,0), do not lie in their domain of definition. For $k > 0$, the system is stable in the large. For $k < 0$, limiting cycles may exist. The decomposition of parameter space is given in Fig. 14.

For type C characteristics, the parameters of auto-oscillation are determined from the following expressions. The period of auto-oscillation $T = 2(\tau_1^* + \tau_2^*)$, where τ_1^* and τ_2^* are defined by the system of Equations (31) and (32). Knowing τ_1^* and τ_2^* we find y_0 and y_1 from the first equations of (23) and (27) after which, determining a_1 and b_1 (also a_2 and b_2) from (22'), we find the motion of the system under auto-oscillation from Equation (22). In the case of relays with type D characteristics, the parameters of auto-oscillation are also found without difficulty.

APPENDIX I

Investigation of the Function Corresponding to (5)

In the investigation of Function (5) we first of all determine the sign of $\frac{dx_1}{dx_0}$ and $\frac{d^2x_1}{dx_0^2}$. Since $\sin \omega\tau$ is positive in the interval $(0, \tau_2)$ and negative in the interval $(\frac{\pi}{\omega}, \tau_1)$, it is easily verified that the numerator of Expression (8), $-\omega(1+\delta)+(1-\delta)(h\sin\omega\tau + \omega\cos\omega\tau)e^{-h\tau}$, is negative in the interval $\left(0, \frac{2\pi}{\omega}\right)$ and that the denominator of Expression (8), $\omega(1-\delta)+(1+\delta)(h\sin\omega\tau - \omega\cos\omega\tau)e^{h\tau}$, is negative in the interval $(0, \tau_2)$ and positive in the interval $\left(\frac{\pi}{\omega}, \tau_1\right)$. Further, it is easily seen that the expression within the square brackets in the numerator of Formula (9), $-2h(1-\delta^2)\sin\omega\tau + \omega(1+\delta)^2 e^{h\tau} - \omega(1-\delta)^2 e^{-h\tau}$, is positive in the interval $\left(0, \frac{2\pi}{\omega}\right)$ since $\omega\sinh\tau > h\sinh\omega\tau$.

It thus follows that $\frac{dx_1}{dx_0} > 0$ in $(0, \tau_2)$ and $\frac{dx_1}{dx_0} < 0$ in $\left(\frac{\pi}{\omega}, \tau_1\right)$ and that the sign of $\frac{d^2x_1}{dx_0^2}$ in the interval $\left(0, \frac{2\pi}{\omega}\right)$ depends on the sign of $\underline{c}$, i.e., in the given case, on the sign of $\underline{k}$: $\frac{d^2x_1}{dx_0^2} > 0$ for $k < 0$ and $\frac{d^2x_1}{dx_0^2} < 0$ for $k > 0$. We then find that, for the "plane" case, the curve $x_1 = x_1(x_0)$ has the asymptote $\left(\text{for } \tau \to \frac{\pi}{\omega}\right)$:

$$x_1 = -e^{-\frac{\pi h}{\omega}} x_0 + \frac{k-2h}{c}\left(1+e^{-\frac{\pi h}{\omega}}\right) - \frac{\delta}{kc}\left(1-e^{-\frac{\pi h}{\omega}}\right).$$

For the "cylindrical" case, the curve $x_1 = x_1(x_0)$ has the asymptote (for $\tau \to 0$) $x_1 = x_0 - 2\delta kc$.

We also note that

$$\frac{dx_1}{dx_0}(\tau_1) = \frac{dx_1}{dx_0}(\tau_2) = \infty.$$

All the above assertions determine the form and orientation of the curve $x_1(x_0)$.

We mention in conclusion that the equation for the half-period, (10), has no solution in the interval $\left(\frac{\pi}{\omega}, \tau_1\right)$. Consequently, on the corresponding Koenig-LeMaire diagram the line $x_1(x_0)$ will not intersect the line $x_1 = -x_0$ (for $\frac{\pi}{\omega} < \tau < \tau_1$).

APPENDIX II

Decomposition of k, h, δ Parameter Space

The boundaries of the topologically similar regions of phase space, for $h < 1$, are determined by the following conditions (see Fig. 3).

1. For $k < 0$, from the condition that unstable limiting cycles appear. This condition determines the boundary of the surface dividing the region where there are no limiting cycles from the region where there exists one unstable limiting "cylindrical" cycle. The corresponding equations are written as: $x_1 = -x_0 = \overline{x}_1$ [$\overline{x}_1$ is determined from Formula (6)] and, in expanded form,

$$\omega(1-\delta)e^{-h\tau} = (\omega\cos\omega\tau - h\sin\omega\tau)(1+\delta),$$

$$(h-k)\sin\omega\tau - \omega\delta\cos\omega\tau = \omega\,\mathrm{sh}\,h\tau + \omega\delta\,\mathrm{ch}\,h\tau.$$

2. For $k > 0$, from the condition that unstable limiting cycles appear (from the condition that the tangent to the curve $x_1(x_0)$ on Fig. 3 be the bisector of the right angle). This condition determines the surface dividing the region in which no limiting cycles exist from the region in which two cycles exist. The corresponding equations take the form:

$$x_1 = -x_0,\ \frac{dx_1}{dx_0} = -1,$$

or, in expanded form,

$$2\omega\delta + (1+\delta)e^{h\tau}(\omega\cos\omega\tau - h\sin\omega\tau) = (1-\delta)e^{-h\tau}(h\sin\omega\tau + \omega\cos\omega\tau),$$

$$(h-k)\sin\omega\tau - \omega\delta\cos\omega\tau = \omega(\mathrm{sh}\,h\tau + \delta\,\mathrm{ch}\,h\tau).$$

3. For $k > 0$, from the condition that the unstable limiting cycle join the boundary of the zone of attraction of the isolated points $(-1,0)$, $(1,0)$. This condition defines the surface dividing the region in which there exist two limiting cycles from the region in which there exists one limiting cycle. The corresponding equations coincide with the equations for $k < 0$: $x_1 = x_0 = \bar{x}_1$.

For $h > 1$, there exists only the continuation of the surface I. Its equations are

$$\omega_1(1-\delta)e^{-h\tau} = (\omega_1\,\mathrm{ch}\,\omega_1\tau - h\,\mathrm{sh}\,\omega_1\tau)(1+\delta),$$

$$(h-k)\,\mathrm{sh}\,\omega_1\tau - \omega_1\delta\,\mathrm{ch}\,\omega_1\tau = \omega_1(\mathrm{sh}\,h\tau + \delta\,\mathrm{ch}\,h\tau),$$

where $\omega_1 = \sqrt{h^2-1}$.

APPENDIX III

Investigation of the Function Corresponding to (13)

For investigating Function (13), we determine the signs of $\frac{dx_1}{dx_0}$ and $\frac{d^2x_1}{dx_0^2}$. As was stated earlier, the time of transfer τ varies within the following intervals (see table below) where τ_1' and τ_2'' are determined from Formulas (16) and (17) :

δ	Motion on "cylinder"	"Plane" motion
$\delta < \delta^*$	$0,\ \tau_1'$	$\tau_1',\ \frac{\pi}{\omega}$
$\delta > \delta^*$	$0,\ \tau_1''$	$\frac{\pi}{\omega},\ \tau_2$

It is easily verified that the numerator of Expression (14) $\omega(1-\delta) - (1+\delta)(h\sin\omega\tau + \omega\cos\omega\tau)e^{-h\tau}$ for $\delta < \delta^*$, is positive in the interval $\left(\tau_1', \frac{\pi}{\omega}\right)$ and negative in the interval $(0, \tau_1')$ but, for $\delta > \delta^*$, is negative in both the intervals $(0, \tau_1'')$ and $\left(\frac{\pi}{\omega}, \tau_2\right)$.

As for the denominator of Expression (14), $-\omega(1+\delta) + (1-\delta)(\omega\cos\omega\tau - h\sin\omega\tau)e^{h\tau}$ it is negative for $\delta < \delta^*$ in the intervals $(0, \tau_2)$ and $\left(\tau_1', \frac{\pi}{\omega}\right)$, negative for $\delta > \delta^*$ in the interval $(0, \tau_1'')$ and positive for $\delta > \delta^*$ in the interval $\left(\frac{\pi}{\omega}, \tau_2\right)$.

We now introduce the notation $f(\tau)$ for the expression in the square brackets in the numerator of Formula (15): $f(\tau) = -2h(1-\delta^2)\sin\omega\tau + \omega(1-\delta)^2e^{h\tau} - \omega(1+\delta)^2e^{-h\tau}$.

We wish to determine the sign of the derivative, $f'(\tau) = h\omega[-2(1-\delta^2)\cos\omega\tau + (1-\delta^2)e^{h\tau} + (1+\delta)^2e^{-h\tau}]$.

We immediately convince ourselves that, for $\delta > 1$, $\cos \omega\tau > 0$ and for $\delta < 1$, $\cos \omega\tau < 0$, the derivative $f'(\tau) > 0$.

We proceed as follows in order to determine the sign of $f'(\tau)$ when $\delta > 1$, $\cos\omega\tau < 0$ and when $\delta < 1$, $\cos \omega\tau > 0$. Let us assume that, in these cases, $f'(\tau) > 0$. Then

$$2(1-\delta^2)\cos\omega\tau < (1-\delta^2)e^{h\tau} + (1+\delta)^2 e^{-h\tau}.$$

Here, the left member of the inequality is positive. It is therefore permissible, without changing the sense of the inequality, to square both members, leading to the expression

$$[(1-\delta)^2 e^{h\tau} - (1+\delta)^2 e^{-h\tau}]^2 + 4(1-\delta^2)^2 \sin^2\omega\tau > 0,$$

the correctness of which is obvious. If now $f'(\tau) < 0$, the sign of the preceding expression would have to be reversed, which is impossible. Thus, $f'(\tau) > 0$ for all τ. In the intervals $(0, \tau_1'')$ and $(\tau_1', \pi/\omega)$ there exists a $\tau = \tau'$, such that $f(\tau') = 0$. It is easily verified that $f(\tau) > 0$ in the intervals $\left(\frac{\pi}{\omega}, \tau_2\right)$, (τ', τ_1''), and $(\tau', \frac{\pi}{\omega})$ and $f(\tau) < 0$ in the intervals $(0, \tau_1')$, $(0, \tau')$ and (τ_1', τ'). It follows from what has been said that $\frac{dx_1}{dx_0} > 0$ in the intervals $(0, \tau_1')$ and $(0, \tau_1'')$, and $\frac{dx_1}{dx_0} < 0$ in the intervals $\left(\tau_1', \frac{\pi}{\omega}\right)$ and $\left(\frac{\pi}{\omega}, \tau_2\right)$.

With the condition that $\delta < \delta^*$ in the intervals $(0, \tau_1')$ and (τ_1', τ'), or for $\delta > \delta^*$ in the interval $(0, \tau')$

$$\frac{d^2x_1}{dx_0^2} > 0, \quad \text{if } k > 0, \text{and} \frac{d^2x_1}{dx_0^2} < 0, \quad \text{if } k < 0.$$

With the condition that $\delta < \delta^*$ in the interval $\left(\tau', \frac{\pi}{\omega}\right)$ or for $\delta > \delta^*$ in the intervals (τ_1', τ_1'') and $\left(\frac{\pi}{\omega}, \tau_2\right)$: $\frac{d^2x_1}{dx_0^2} > 0$ if $k < 0$, and $\frac{d^2x_1}{dx_0^2} < 0$ if $k > 0$.

Incidentally, we may mention that

$$\frac{dx_1}{dx_0}(\tau_1') = 0, \qquad \frac{dx_1}{dx_0}(\tau_1') = \infty, \qquad \frac{dx_1}{dx_0}(\tau_2) = \infty.$$

The curve $x_1 = x_1(x_0)$ has the asymptote (for $\tau_1 \to 0$)

$$x_1 = x_0 - \frac{2\delta}{kc}$$

and the asymptote (for $\tau \to \frac{\pi}{\omega}$)

$$x_1 = -e^{-\frac{\pi h}{\omega}} x_0 + \frac{2h-k}{c}\left(1 + e^{-\frac{\pi h}{\omega}}\right) - \frac{\delta}{kc}\left(1 - e^{-\frac{\pi h}{\omega}}\right).$$

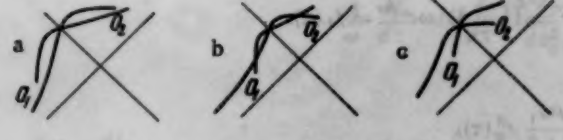

Fig. 15.

It is necessary to consider in greater detail the case $k > 0$, $\delta > \delta^*$ (Fig. 8). In this case, a variation in the system parameters may occasion a reversal of the stability of the limiting cycles. In order to follow this out in detail, it is necessary to consider not only the point transformation $x_1 = x_1(x_0)$, but also the inverse transformation, $x_0' = x_0'(x_1')$.

In this case, the corresponding function has the form:

$$x_0' = \frac{(1-\delta)\cos\omega\tau - (1+\delta)e^{-h\tau}}{c\sin\omega\tau}\omega + \delta - \frac{(1-\delta)(h-k)}{c},$$

$$x_1' = \frac{-(1+\delta)\cos\omega\tau + (1-\delta)\,e^{h\tau}}{c\sin\omega\tau}\,\omega - \delta - \frac{(1+\delta)(h-k)}{c}.$$

As a result, the diagram shows the curve $x_1' = x_1'(x_0)$ as symmetric to the curve $x_1 = x_1(x_0)$ with respect to the line $x_1 = -x_0$. The different cases which might arise here are shown in Fig. 15 (cases a, b, c).

It is necessary to show that the curves of Fig. 15 are not intersected in more than three points. Additional points of intersection, over and above those shown in Fig. 15, may occur in sections where both curves simultaneously are convex upward. Consequently, it is sufficient to consider the interval $(\tau_0, \bar{\tau})$ of variation of τ, in which the second derivatives of $x_1(x_0)$ and $x_1'(x_0)$ with respect to x_0 are negative. τ_0 is the value of τ for which

$$\frac{d^2x_1}{dx_0^2} = \frac{d^2x_1'}{dx_0^2} = 0,$$

and $\bar{\tau}$, as above, is determined from the condition of contact:

$$-\omega(1+\delta) + (1-\delta)(-h\sin\omega\tau + \omega\cos\omega\tau)\,e^{h\tau} = 0.$$

We now consider the curve $x(x_0) = x_1'(x_0) - x_1(x_0)$. If $x(x_0) = 0$ at not more than three points, and consequently, if $\frac{dx_1}{dx_0} - = 0$ at not more than two points, then the curves in Fig. 11 are intersected in not more than three points. We now show that the curves $f_1(x_0) = \frac{dx_1}{dx_0}$ and $f_2(x_0) = \frac{dx_1'}{dx_0}$ intersect no more than twice in $(\tau_0, \bar{\tau})$. Such will be the case if, in $(\tau_0, \bar{\tau})$, $\frac{d^2f_1}{dx_0^2} > 0$ and $\frac{d^2f_2}{dx_0^2} < 0$.

We show that $\frac{d^2f_1}{dx_0^2} > 0$ in $(\tau_0, \bar{\tau})$.

We have

$$\frac{d^2f_1}{dx_0^2} = \frac{d}{d\tau}\left(\frac{d^2x_1}{dx_0^2}\right)\frac{d\tau}{dx_0}.$$

But, since

$$\frac{dx_0}{d\tau} = \frac{-\omega(1+\delta) + (1-\delta)(-h\sin\omega\tau + \omega\cos\omega\tau)\,e^{h\tau}}{a\sin^2\omega\tau}\,\omega < 0$$

in $(0, \bar{\tau})$, it is necessary that the expression $\frac{d}{d\tau}\left(\frac{d^2x_1}{dx_0^2}\right)$ be less than zero in $(\tau_0, \bar{\tau})$.

We introduce the notation

$$\frac{-\omega(1+\delta) + (1-\delta)(-h\sin\omega\tau + \omega\cos\omega\tau)\,e^{h\tau}}{\sin\omega\tau} = f_3(\tau),$$

$$-2(1-\delta^2)\,h\sin\omega\tau + (1-\delta)^2\,\omega e^{h\tau} - (1+\delta)^2\,\omega e^{-h\tau} = f_4(\tau).$$

With this,

$$\frac{a}{\omega}f_4(\tau) = \frac{d^2x_1}{dx_0^2}f_3^2(\tau),$$

$$\frac{a}{\omega}\frac{df_4(\tau)}{d\tau} = \frac{d}{d\tau}\left(\frac{d^2x_1}{dx_0^2}\right)f_3^2(\tau) + \frac{d^2x_1}{dx_0^2}\,3f_3^2\,\frac{df_3}{d\tau}. \qquad (A)$$

Here, $\frac{df_4}{d\tau} > 0$ for all τ, since $f_4(\tau)$ coincides with $f(\tau)$ (see this Appendix, above) $f_3 < 0$ in $(0, \bar{\tau})$

and $\frac{d^2x_1}{dx_0^2} < 0$ in $(\tau_0, \bar{\tau})$.

We consider the derivative

$$\frac{df_3}{d\tau} = \frac{\omega^2(1+\delta)\cos\omega\tau - (1-\delta)(\sin^2\omega\tau - h\omega\sin\omega\tau\cos\omega\tau + \omega^2\cos^2\omega\tau)\,e^{h\tau}}{\sin^2\omega\tau}.$$

With this, $\frac{df_3}{d\tau} > 0$ in $\left(0, \frac{\pi}{2\omega}\right)$ and in part of the interval $\left(\frac{\pi}{2\omega}, \frac{\pi}{\omega}\right)$. In $\left(\frac{\pi}{2\omega}, \frac{\pi}{\omega}\right)$, $\frac{df_3}{d\tau}$ may be less than zero. From $\frac{df_3}{d\tau} = 0$ follows

$$\frac{1+\delta}{1-\delta} = \frac{e^{h\tau}}{\omega^2}\,\frac{\sin^2\omega\tau - h\omega\sin\omega\tau\cos\omega\tau + \omega^2\cos^2\omega\tau}{\cos\omega\tau}.$$

Substituting the expression for $\frac{1+\delta}{1-\delta}$ in $\frac{d^2x_1}{dx_0^2}$, we obtain

$$\frac{d^2x_1}{dx_0^2} > 0, \quad \text{i. e., in } (\tau_0, \bar{\tau}), \frac{df_3}{d\tau} > 0,$$

because in $(\tau_0, \bar{\tau})$, $\frac{d^2x_1}{dx_0^2} < 0$. Consequently, from (A) $\frac{d}{d\tau}\left(\frac{d^2x_1}{dx_0^2}\right) < 0$ in $(\tau_0, \bar{\tau})$.

We now show that $\frac{d^2f_3}{dx_0^2} < 0$ in (τ_0, τ).

We have

$$\frac{d^2f_3}{dx_0^2} = \frac{d}{d\tau}\left(\frac{d^2x_1'}{dx_0^2}\right)\frac{d\tau}{dx_0}.$$

But, since

$$\frac{dx_0}{d\tau} = \frac{dx_0'}{d\tau} = \frac{-\omega(1-\delta) + (1+\delta)(h\sin\omega\tau + \omega\cos\omega\tau)\,e^{-h\tau}}{a\sin^2\omega\tau}\,\omega > 0$$

in $\left(0, \frac{\pi}{\omega}\right)$, it is necessary that the expression $\frac{d}{d\tau}\left(\frac{d^2x_1'}{dx_0^2}\right)$ be less than zero in $(\tau_0, \bar{\tau})$. We introduce the notation

$$f_5(\tau) = \frac{-\omega(1-\delta) + (1+\delta)(h\sin\omega\tau + \omega\cos\omega\tau)\,e^{-h\tau}}{\sin\omega\tau}.$$

Then

$$-\frac{a}{\omega}f_4(\tau) = \frac{d^2x_1'}{dx_0^2}f_5^3(\tau),$$

$$-\frac{a}{\omega}\frac{df_4}{d\tau} = \frac{d}{d\tau}\left(\frac{d^2x_1'}{dx_0^2}\right)f_5^3 + \frac{d^2x_1'}{dx_0^2}3f_5^2\frac{df_5}{d\tau}. \qquad \text{(B)}$$

But, $\frac{df_4}{d\tau} > 0$ for all τ, $f_5 > 0$ in $\left(0, \frac{\pi}{\omega}\right)$, and $\frac{d^2x_1'}{dx_0^2} < 0$ in $(\tau_0, \bar{\tau})$.

We consider the expression

$$\frac{df_5}{d\tau} = \frac{\omega^2(1-\delta)\cos\omega\tau - (1+\delta)\,e^{-h\tau}(\sin^2\omega\tau + h\omega\sin\omega\tau\cos\omega\tau + \omega^2\cos^2\omega\tau)}{\sin^2\omega\tau}.$$

Here $\frac{df_5}{d\tau} < 0$ in $(0, \tau^*)$, where $\tau^* > \frac{\pi}{2\omega}$.

From $\frac{df_5}{d\tau}(\tau^*) = 0$ it follows that

$$\frac{1-\delta}{1+\delta} = \frac{e^{-h\tau}}{\omega^2} \frac{\sin^2\omega\tau + h\omega\sin\omega\tau\cos\omega\tau + \omega^2\cos^2\omega\tau}{\cos\omega\tau}.$$

Substituting the expression for $\frac{1-\delta}{1+\delta}$ in $\frac{dx_1'}{dx_0}$, we get $\frac{dx_1'}{dx_0} < 0$, whence $\tau^* > \bar{\tau}$, i.e., in $(\tau_0, \bar{\tau})$, $\frac{df_5}{d\tau} < 0$ and, from (B),

$$\frac{d}{d\tau}\left(\frac{d^2x_1'}{dx_0^2}\right) < 0.$$

In case a (Fig. 15) there is a stable limiting cycle. At the moment of transfer, at point O_1 via the curve $x_1' = x_1'(x_0)$ and at point O_2 via $x_1 = x_1(x_0)$, there appear two unstable cycles, between which are found stable ones. Further, with variation of the system parameters, these unstable cycles approximate to the stable ones and fuse with them. As a result, there appears but one unstable limiting cycle.

APPENDIX IV

Investigation of the Function Corresponding to (18)

For the investigation of the function corresponding to (18), we determine the signs of $\frac{dx_1}{dx_0}$ and $\frac{d^2x_1}{dx_0^2}$. We designate by $f_1(\tau)$ the numerator of Expression (19), $f_1(\tau) = \omega_1(1-\delta) - (1+\delta)(\omega_1 \operatorname{ch}\omega_1\tau + h \operatorname{sh}\omega_1\tau)\, e^{-h\tau}$, and thus find that $f_1'(\tau) > 0$ for any positive τ. Let $f_1(\tau_1) = 0$ for $\tau = \tau_1$. Then $f_1 < 0$ for $0 < \tau < \tau_1$, and $f_1 > 0$ for $\tau_1 < \tau < \infty$.

Let us designate by $f_2(\tau)$ the denominator of Expression (19). It is easily seen that $f_2(0) < 0$, $f_2'(\tau) < 0$ and, consequently, $f_2(\tau) < 0$ for $0 < \tau < \infty$.

Further, in Formula (20) let

$$f(\tau) = -2h(1-\delta^2) \operatorname{sh}\omega_1\tau + \omega_1(1-\delta)^2 e^{h\tau} - \omega_1(1+\delta)^2 e^{-h\tau}.$$

We then find that

$$f(0) < 0, \quad f(\tau_1) < 0, \quad f(\infty) = \infty, \quad f'(0) > 0$$

and, moreover, $f'(\tau) = 0$ for τ satisfying the relationships $\frac{1+\delta}{1-\delta} = e^{(h-\omega_1)\tau'}$ and $\frac{1+\delta}{1-\delta} = e^{(h+\omega_1)\tau''}$, where $\tau'' < \tau'$. Since $f(\tau'') = (1-\delta)^2(h+\omega_1)(1-e^{2\omega_1\tau''}) < 0$, $f(\tau)$ consequently changes sign only once, for some $\tau^0 > \tau_1$, i.e., in the case of "plane" motion. From this, it is not difficult to see that

$$\frac{dx_1}{dx_0} > 0 \text{ for } 0 < \tau < \tau_1, \qquad \frac{px_1}{dx_0} < 0 \text{ for } \tau_1 < \tau < \infty,$$

$$\frac{d^2x_1}{dx_0^2} > 0 \begin{cases} \text{for } 0 < k < h-\omega_1 \text{ or } k > h+\omega_1 \text{ in interval } (0, \tau_1), (\tau_1, \tau^0), \\ \text{for } k < 0 \text{ or } h-\omega_1 < h < h+\omega_1 \text{ in interval } (\tau^0, \infty), \end{cases}$$

$$\frac{d^2x_1}{dx_0^2} < 0 \begin{cases} \text{for } 0 < k < h-\omega_1 \text{ or } k > h+\omega_1 \text{ in interval } (\tau^0, \infty), \\ \text{for } k < 0 \text{ or } h-\omega_1 < h < h+\omega_1 \text{ in interval } (0,\tau_1)(\tau_1, \tau^0). \end{cases}$$

The curve $x_1(x_0)$ has (for $\tau \to \infty$) the asymptote

$$x_1 = \frac{h - k - \frac{\delta}{k} - \omega_1 + \delta(\omega_1 + h)}{c}.$$

For $\tau \to 0$ the asymptote is

$$x_1 = x_0 - \frac{2\delta}{kc}.$$

APPENDIX V

Decomposition of Parameter Space

For $h < 1$, the conditions defining the boundaries of the topologically similar regions of phase space will be the following (Fig. 8).

1. The condition of transfer of a stable limiting cycle from the "cylinder" to the "plane" or conversely, gives the equation for the boundary of the surface S_1: $x_1 = -x_0 = -\bar{x}_0$ (here, $\bar{x}_0 = \delta - \frac{k(1+\delta)}{c}$ or, in expanded (parametric) form

$$(1+\delta)(\omega\cos\omega\tau + h\sin\omega\tau) = (1-\delta)\omega e^{h\tau},$$

$$\omega(1-\delta)\cos\omega\tau + (\delta h - h + 2k)\sin\omega\tau = \omega(1+\delta)e^{-h\tau} \quad (\omega\tau \leqslant \pi).$$

2. The condition that two unstable limiting cycles appear on the "cylinder" gives the equation for the surface S_2: $x_1 = x_1' = \bar{x}_1$, $x_0 = x_0'$ (here, $\bar{x}_1 = -\delta - \frac{k(1-\delta)}{c}$) or, in expanded form:

$$\omega(1+\delta)e^{-h\tau_1} = (1-\delta)(-h\sin\omega\tau_1 + \omega\cos\omega\tau_1), \qquad (\omega\tau_1 \leqslant \pi, \quad \omega\tau_2 \leqslant \pi),$$

$$\omega(1+\delta)\cos\omega\tau_2 + (h + \delta h - 2k)\sin\omega\tau_2 = (1-\delta)\omega e^{h\tau_2},$$

$$\frac{(1+\delta)\cos\omega\tau_1 - (1-\delta)e^{h\tau_1}}{\sin\omega\tau_1}\omega = 2(k-h) + \frac{(1-\delta)\cos\omega\tau_2 - (1+\delta)e^{-h\tau_2}}{\sin\omega\tau_2}\omega.$$

3. The condition that three cycles on the "cylinder" merge into one unstable one gives the equation for surface S_3: $x_1 = -x_0$, $\frac{dx_1}{dx_0} = 1$ or, in expanded form,

$$2\delta\omega\cos\omega\tau + 2(h-k)\sin\omega\tau = \omega[(1-\delta)e^{h\tau} - (1+\delta)e^{-h\tau}], \quad (\omega\tau \leqslant \pi)$$

$$2\omega - (1+\delta)(h\sin\omega\tau + \omega\cos\omega\tau)e^{-h\tau} = (1-\delta)(-h\sin\omega\tau + \omega\cos\omega\tau)e^{h\tau}.$$

4. The condition for the disappearance of an unstable limiting cycle from the "cylinder" gives the equation for surface S_4: $x_1 = -x_0 = \bar{x}_1$ or, in expanded form,

$$(1-\delta)(-h\sin\omega\tau + \omega\cos\omega\tau) = (1+\delta)\omega e^{-h\tau} \quad (\omega\tau \leqslant \pi),$$

$$\omega(1+\delta)\cos\omega\tau + (\delta h + h - 2k)\sin\omega\tau = \omega(1-\delta)e^{h\tau}.$$

5. The condition for the emergence of an unstable "plane" limiting cycle gives the equation for surface S_5: $x_1 = -x_0$, $\frac{dx_1}{dx_0} = -1$ or, in expanded form,

$$2\delta\omega\cos\omega\tau + 2(h-k)\sin\omega\tau = \omega[(1-\delta)e^{h\tau} - (1+\delta)e^{-h\tau}] \quad (\pi \leqslant \omega\tau < 2\pi).$$

$$2\omega\delta + (1+\delta)(h\sin\omega\tau + \omega\cos\omega\tau)e^{-h\tau} = (1-\delta)(\omega\cos\omega\tau - h\sin\omega\tau)e^{h\tau},$$

6. The condition that an unstable "plane" cycle degenerate to the boundary of the zone of attraction of the isolated points $(-1,0)$ and $(1,0)$ gives the equation for surface S_6: $x_1 = -x_0 = \bar{x}_1$, coinciding with the equation for surface S_4.

In the equations for surface S_8, $\omega\tau$ varies in the interval ($\pi \leq \omega\tau < 2\pi$). For h > 1, there exist (Fig. 9) only the extensions of surface S_1

$$(1+\delta)(\omega_1 \operatorname{ch}\omega_1\tau + h\operatorname{sh}\omega_1\tau) = (1-\delta)\omega_1 e^{h\tau},$$
$$\omega_1(1-\delta)\operatorname{ch}\omega_1\tau + (\delta h - h + 2k)\operatorname{sh}\omega_1\tau = (1+\delta)\omega_1 e^{-h\tau}$$

and the plane $\delta = 1$ divides the region devoid of limiting cycles from the region in which there exists one limiting cycle ("plane" or "cylindrical").

APPENDIX VI

Construction of Parameter Space Decomposition

To construct the decomposition of parameter space into regions of qualitatively identical dynamic system behavior, we subtract Equation (32) from Equation (31). The result is

$$\frac{1-\delta}{\delta}\,\frac{\cos\omega\tau_1 - \operatorname{ch} h\tau_1}{\sin\omega\tau_1} = \frac{\cos\omega\tau_2 + \operatorname{ch} h\tau_2}{\sin\omega\tau_2}. \qquad (a)$$

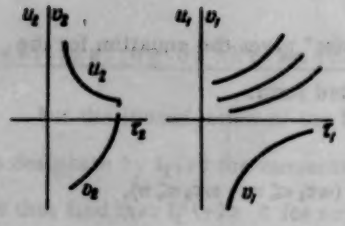

Fig. 16.

Equations (33) and (a) may be written in the form

$$u_2(h, \tau_2) = \frac{1-\delta}{\delta}u_1(h, \tau_1), \qquad (b)$$

$$v_2(h, \tau_2) = v_1(h, \tau_1). \qquad (c)$$

The limits of variation of τ_1 and τ_2 were stated above. We fix h, and construct the curves corresponding to the left and right members of Equations (b) and (c) for various $\alpha = \frac{1-\delta}{\delta}$ (Fig. 16). Let the curves $u_2(\tau_2)$ and $v_2(\tau_2)$ be plotted on transparent paper. Superimposing the plane $u_2(v_2)$, τ_2 on the plane $u_1(v_1)$, τ_1, we slide the planes along the axis of abscissas seeking the points of intersection of the constructed curves. The system of Equations (b) and (c) will have solutions τ_1^0 and τ_2^0 if the points of intersection of the curves u_2 with αu_1 and v_2 with v_1 lie on one vertical. Determining τ_1^0 and τ_2^0 from Equation (a), we find the points which lie on the surface dividing, in δ, k, h space, the region in which two limiting cycles exist from the region where no limiting cycles exist.

For finding the boundaries dividing the region in which two cycles exist from the region in which one cycle exists, we take Equation (34), with h fixed, in the form

$$\tau_1 = \bar{\tau}_1, \quad u_2(\tau_2) = \alpha u_1(\bar{\tau}_1)$$

and, correspondingly, Equation (35) in the form

$$\tau_2 = \bar{\tau}_2, \quad u_2(\bar{\tau}_2) = \alpha u_1(\tau_1).$$

Then, in a manner analogous to that given above, we find the points of the bounding surfaces.

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DETERMINATION OF PARAMETERS FOR CORRECTIVE DEVICES IN LINEAR SERVOSYSTEMS USING GIVEN GENERALIZED PARAMETERS

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An analytical method is described for determination of parameters for correcting devices in a linear servosystem when the values of some of the generalized parameters of the system are given, such as: the asymptotic crossover frequency, the phase margin of stability and the relative quality factor. A table of basic formulas for calculations are given, and examples of computations are carried out for series and parallel correcting devices and also for their combinations.

INTRODUCTION

In the present paper a procedure is given for selection and determination of parameters for correcting devices, which makes it possible to solve the second stage of the synthesis problem for servosystems as formulated, for instance, in [1]. The problem treated in the present paper, however, differs from the problem of [1]; here it is not an over-all optimal characteristic of the entire system which is given, but only optimal values of some of the most important generalized parameters which affect the precision of the system. In servosystems, which have typical logarithmic amplitude characteristics (l. a. c.), such parameters are: the asymptotic crossover frequency ω_c, the phase margin of stability φ_c, and the relative quality factor $D_c = \frac{D}{\omega_c}$ (D is the quality factor, or velocity amplification factor of an astatic system of the first order). Thus, the problem consists in determining the transfer function and the parameters of a correcting arrangement, when the following are given: the shape of the typical l.a.c., the transfer function of a certain part in the block diagram of the servosystem whose components have been selected for technical reasons, and the numerical values of ω_c, φ_c, and D_c. Let us emphasize that ω_c, φ_c, and D_c in turn are selected in such a way that at the given magnitudes and character of the input pulse, the over-all error of the system should be at a minimum.

When selecting one of the four types of l.a.c. [1] it is necessary to take into account the following:

1. Types I and III of the l.a.c. are the simplest for practical realization;

2. The error in a system having a l.a.c. of type II or type IV can be made smaller in magnitude than for a system with a l.a.c. of type I or type III;

3. In the presence of perturbation input only, the smallest error can be obtained with a system having a l.a.c. of type I;

4. If the system is under the influence of both controlling signals and perturbations, and the relative quality factor is large, with l.a.c. of types II and IV a somewhat smaller minimal error can be obtained than with types I and III.

The block diagram of a servosystem with series and parallel correcting devices usually has the form shown in Fig. 1.

The transfer function of the open circuit loop for a servosystem of the above-described kind, as is known, is

$$K(p) = \frac{K_k(p)\,K_u(p)}{1 + K_{oc}(p)}, \tag{1}$$

where $K_u(p) = K_1(p)\,K_2(p)$ is the transfer function of the open circuit loop of the given (uncorrected) part of the system, $K_k(p)$ is the transfer function of the series corrective device, $K_{oc}(p) = K_2(p)\,K'_k(p)$ is the open circuit transfer function of the local feedback loop, $K'_k(p)$ is the transfer function of the parallel correcting device, $K_1(p)$ is the transfer function of the system links not encompassed by the local feedback loop and $K_2(p)$ the transfer function of the system links included in the local feedback loop.

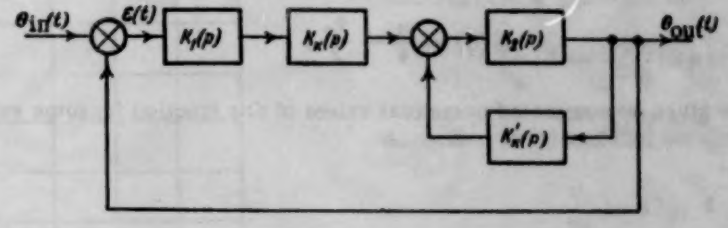

Fig. 1.

It is necessary to select $K_k(p)$ and $K'_k(p)$ in such a way that the over-all open circuit transfer function of the system

$$K(p) = K_i(p), \tag{2}$$

where ω_c, φ_c, and D_c have the chosen optimal values.

Here $K_i(p)$ is an open circuit transfer function for the system, which has a l.a.c. of either type I, II, III, or IV of those given in paper [2]; thus i = I, II, III, or IV.

In special cases

$$K(p) = K_k(p)\,K_u(p), \tag{3}$$

when only series correcting devices are used in the system, and

$$K(p) = \frac{K_u(p)}{1 + K_{oc}(p)} \tag{4}$$

when only parallel correcting devices are used.

1. Basic Relationships for Computations

To find the basic relationships needed for this computation let us investigate the region of each of the typical l.a.c. [2] in the neighborhood of the asymptotic crossover frequency ω_c. This will permit us to find the connection between the parameters which are included directly in the open circuit transfer function and the generalized parameters (ω_c, φ_c, and D_c). Approximation equations connecting the two sets of parameters are as follows:

for a l.a.c. of type I:

$$\frac{\omega_3}{\omega_c}\left(1 - \frac{\omega_c}{D}\right) + \frac{\omega_c}{\omega_3} \approx 2\beta(\varphi),$$

for a l.a.c. of type II:

$$2\frac{\omega_3}{\omega_c}\left(1 - \sqrt{\frac{\omega_c}{D}}\right) + \frac{\omega_c}{\omega_3} \approx 2\beta(\varphi), \tag{5}$$

for a l.a.c. of type III:

$$\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_c}{D}\right)+2\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi),$$

for a l.a.c. of type IV:

$$\frac{\omega_2}{\omega_c}\left(1-\sqrt{\frac{\omega_c}{D}}\right)+\frac{\omega_c}{\omega_3}\approx\beta(\varphi).$$

Here $\beta(\varphi)$ is a linear function of φ_c with a negative slope:

$$\beta(\varphi)=\frac{\pi}{4}-\frac{\varphi_c}{2}. \tag{6}$$

In Table 1 there are given approximated numerical values of this function for some values of the phase margin of stability φ_c.

TABLE 1

φ_c, deg.	20	25	30	35	40	45	50	55	60
$\beta(\varphi)$	0.60	0.56	0.52	0.48	0.44	0.39	0.35	0.3	0.26

Deduction of Equations (5) is given in an appendix to this paper.

For a servosystem with large relative quality factor $\left(\sqrt{\frac{D}{\omega_c}}\gg 1\right)$ Equations (5) are simplified to:

$$\frac{\omega_2}{\omega_c}+\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi),\quad 2\frac{\omega_2}{\omega_c}+\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi),$$

$$\frac{\omega_2}{\omega_c}+2\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi),\quad \frac{\omega_2}{\omega_c}+\frac{\omega_c}{\omega_3}\approx\beta(\varphi). \tag{7}$$

To solve equations of the type of (5) or (7), it is necessary to have an additional condition given. This condition can be, for instance, that the ratio ω_3/ω_2 for a given φ_c shall have a certain minimum value, or in other words, that the frequency band width of the l.a.c. with a slope of 6 db per octave (or 20 db per decade) should have a given minimum value in the neighborhood of the asymptotic crossover frequency φ_c. In this case the servosystem will have a slowly changing phase-frequency characteristic of the open circuit loop near the crossover frequency (Fig. 2), which will assure the constancy of φ_c, one of the basic parameters of the system when the amplification factor of the system is changed. The latter is important from the point of view of maintenance needed during operation of the system. In addition, the fulfillment of this condition is also of practical importance in the following respect: when correcting the characteristics of the servosystem, for instance, by means of derivative loops, the quantity ω_3/ω_2 is numerically equal to the loss in the quality factor of the system caused by the introduction of the correcting loops.

Let us solve Equations (5) for the l.a.c. of type I. The relationships needed for the other types of l.a.c. can be obtained from the corresponding Equations (5) in a similar manner. Therefore, the results for l.a.c. of types II, III, and IV will be given below without proof. On the basis of (5) for a l.a.c. of type I, we have:

$$\frac{\omega_3}{\omega_2}\approx\frac{1-\frac{\omega_c}{D}}{2\beta^2(\varphi)-B(\omega)}, \tag{8}$$

where

$$B(\omega)=\left[\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_c}{D}\right)\right]^2+\left(\frac{\omega_c}{\omega_3}\right)^2. \tag{9}$$

Because $B(\omega) > 0$, the minimum of $\frac{\omega_3}{\omega_2}$ at a fixed value of φ_c, as is apparent from (8), will occur when $B(\omega)$ is at a minimum. From Equation (9), after substitution of $\frac{\omega_c}{\omega_3}=2\beta(\varphi)-\frac{\omega_2}{\omega_c}\cdot\left(1-\frac{\omega_c}{D}\right)$ from (5), it follows that $B(\omega)$ has a minimum at

$$\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_c}{D}\right)=\beta(\varphi). \tag{10}$$

According to (5) and (10) we obtain

$$\frac{\omega_c}{\omega_3}=\beta(\varphi). \tag{11}$$

Thus, the minimum of $\frac{\omega_3}{\omega_2}$ occurs when the components in the left part of Equation (5) are equal. For l.a.c. of type I (see Equation (45) in the appendix) we have

$$\frac{\omega_1}{\omega_3}\approx\frac{\omega_c}{D}. \tag{12}$$

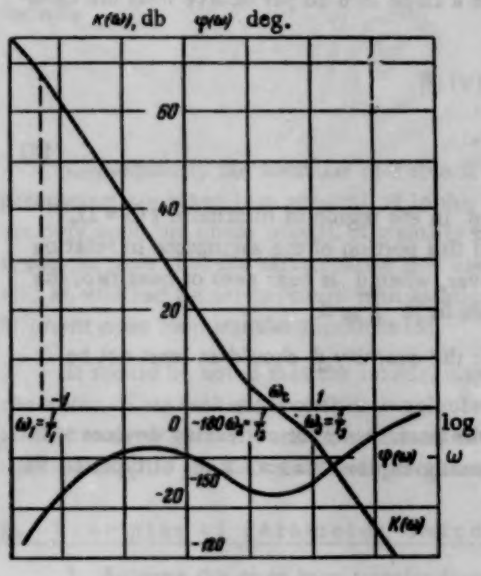

Fig. 2

Equations (10) to (12) establish for l.a.c. of type I the required relationship between the relative conjugated frequencies φ_c and D_c at the minimum value of the frequency interval $\omega_2 - \omega_3$. Hence:

$$\frac{\omega_2}{\omega_c}\approx\frac{\beta(\varphi)}{1-\frac{\omega_c}{D}},\quad \frac{\omega_3}{\omega_c}\approx\frac{1}{\beta(\varphi)},\quad \frac{\omega_1}{\omega_c}\approx\frac{\omega_c}{D}\frac{\beta(\varphi)}{1-\frac{\omega_c}{D}}=\frac{\beta(\varphi)}{\frac{D}{\omega_c}-1}. \tag{13}$$

In the process of designing servosystems there may also be initially given one of the conjugated frequencies; but because the margin of phase stability in a system must in any case not be lower than 20°, it is not always possible to fulfill the above indicated minimum condition for the ratio ω_3/ω_2. In this case Formulas (13) take on a somewhat different form.

Let

$$\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_c}{D}\right)=d\beta(\varphi). \tag{14}$$

Then from Equation (5) for l.a.c. of type I we have

$$\frac{\omega_c}{\omega_3}=(2-d)\beta(\varphi). \tag{15}$$

The relative conjugating frequencies are substantially positive quantities; therefore, Equations (14) and (15) make sense only provided the quantity d satisfies the inequality

$$0<d<2. \tag{16}$$

At $d = 1$ the condition for the ratio ω_3/ω_2 to become a minimum is fulfilled. According to Equations (12), (14), and (15), we finally arrive at:

$$\frac{\omega_3}{\omega_c} \approx \frac{d\beta(\varphi)}{1-\frac{\omega_c}{D}}, \quad \frac{\omega_3}{\omega_c} \approx \frac{1}{(2-d)\beta(\varphi)}, \quad \frac{\omega_1}{\omega_c} \approx \frac{d\beta(\varphi)}{\frac{D}{\omega_c}-1}. \tag{17}$$

Usually it is desirable to strive for fulfillment of the condition of having the smallest possible frequency band width in the l.a.c. with a slope of 6 db per octave near the crossover frequency at a prescribed margin of stability, that is to have d = 1. However, if $d \neq 1$, but has one of the values possible according to (16), it is easy to estimate how much the length of the asymptotic portion with a slope of 6 db per octave near the crossover frequency is increased.

On the basis of (13) and (17) we have:

$$l = \frac{\omega_3/\omega_2}{(\omega_3/\omega_2)_{\min}} = \frac{1}{d(2-d)}. \tag{18}$$

An analysis of Formula (18) shows that l changes slowly with d in the region of minimum l (d = 1). For instance, for the interval $0.6 \leq d \leq 1.4$, the increase in length of this portion of the asymptote in relation to its minimum length is not more than 19% ($1 \leq l \leq 1.19$). However, when d is near zero or near two, the quantity l begins to grow rapidly. At $0 < d \leq 0.3$ and $1.7 \leq d < 2$ we have $l \geq 2$.

Consequently, when designing servosystems, it is desirable that the quantity d should at least not be outside the range $0.4 < d < 1.6$.

The basic relationships used henceforth for determination of the parameters for correcting devices with l.a.c. of type I are given in Formulas (13) and (17) above. Corresponding expressions for l.a.c. of types II, III, and IV are presented in Table 2.

TABLE 2

Parameter	Type of l. a. c. I	II	III	IV
$\frac{\omega_2}{\omega_c}$	$\frac{\beta(\varphi)}{1-\frac{\omega_c}{D}}$	$\frac{1}{2}\frac{\beta(\varphi)}{1-\sqrt{\frac{\omega_c}{D}}}$	$\frac{\beta(\varphi)}{1-\frac{\omega_c}{D}}$	$\frac{1}{2}\frac{\beta(\varphi)}{1-\sqrt{\frac{\omega_c}{D}}}$
$\frac{\omega_3}{\omega_c}$	$\frac{1}{\beta(\varphi)}$	$\frac{1}{\beta(\varphi)}$	$\frac{2}{\beta(\varphi)}$	$\frac{2}{\beta(\varphi)}$
$\frac{\omega_1}{\omega_c}$	$\frac{\beta(\varphi)}{D_c-1}$	$\frac{1}{2}\frac{\beta(\varphi)}{\sqrt{D_c}-1}$	$\frac{\beta(\varphi)}{D_c-1}$	$\frac{1}{2}\frac{\beta(\varphi)}{\sqrt{D_c}-1}$
$\frac{\omega_2}{\omega_c}$ for $d \neq 1$	$\frac{d\beta(\varphi)}{1-\frac{1}{D_c}}$	$\frac{d}{2}\frac{\beta(\varphi)}{1-\sqrt{\frac{1}{D_c}}}$	$\frac{d\beta(\varphi)}{1-\frac{1}{D_c}}$	$\frac{d}{2}\frac{\beta(\varphi)}{1-\sqrt{\frac{1}{D_c}}}$
$\frac{\omega_3}{\omega_c}$ for $d \neq 1$	$\frac{1}{(2-d)\beta(\varphi)}$	$\frac{1}{(2-d)\beta(\varphi)}$	$\frac{2}{(2-d)\beta(\varphi)}$	$\frac{2}{(2-d)\beta(\varphi)}$
$\frac{\omega_1}{\omega_c}$ for $d \neq 1$	$\frac{d\beta(\varphi)}{D_c-1}$	$\frac{d}{2}\frac{\beta(\varphi)}{\sqrt{D_c}-1}$	$\frac{d\beta(\varphi)}{D_c-1}$	$\frac{d}{2}\frac{\beta(\varphi)}{\sqrt{D_c}-1}$

If the system includes links which have inertia and whose time constant $T_i \ll \frac{1}{\omega_c}$ (small parameters), then the open loop transfer function of the System (2) can be represented as follows:

$$K(p) = K_1(p) \frac{1}{\prod_{i=4}^{n} (1 + pT_i)}. \tag{19}$$

Taking into consideration the small parameters, as shown in the appendix, the equations that connect the relative conjugated frequencies and generalized parameters have exactly the same form as Equations (5) except that in the right side instead of $\beta(\varphi)$ there must be substituted $\beta_n(\varphi)$, which is determined by the formula

$$\beta_n(\varphi) = \beta(\varphi) - \frac{1}{2}\omega_c \sum_{i=4}^{n} T_i. \tag{20}$$

Consequently the formulas of Table 2 can be used also for systems having a typical l.a.c., when small parameters are taken into account, if in the formulas of this table $\beta(\varphi)$ is replaced by $\beta_n(\varphi)$, which depends not only upon the phase margin of stability but also upon the small parameters of the system. In addition, it is easy to show that the relationships that are already obtained, or that will be found in a similar manner, can also be utilized for servosystems with astaticism of the second order and can also be extended to systems with different open loop transfer functions [2].

It should be noted that the relationship obtained, as is evident from Table 2, are of a simple algebraic character. This makes it possible to calculate the parameters for correcting devices quickly, without laborious graphic computations. Besides, by means of Condition (16) it is possible to judge whether or not the designed system could be put into practical operation.

2. Examples of Parameter Calculations for Correcting Devices

1. Suppose the open loop transfer function of the given (uncorrected) part of the system is

$$K_u(p) = \frac{D_u}{p} \frac{1}{(1 + pT_{em})(1 + pT_k)} \frac{1}{\prod_{i=4}^{n} (1 + pT_i)}, \tag{21}$$

where T_{em} is the time constant of the electromechanical components, T_k is the time constant of the amplifying and transforming device, D_u is the amplification factor.

Further let us assume that $T_{em} > T_k$. It is necessary to choose the correcting arrangement in such a way that the over-all transfer function of the system shall correspond to a l.a.c. of type I and have the required values of ω_c, φ_c, and D. Thus, for a l.a.c. of type I [2]:

$$K_I(p) = \frac{D}{p} \frac{1 + pT_2}{(1 + pT_1)(1 + pT_3)}, \tag{22}$$

it is expedient to use for the solution of this problem an integro-differential passive electrical network as shown in Fig. 3, and an inertialess amplifier.

Indeed, assuming that

$$K_k(p) = \frac{D}{D_u} \frac{(1 + pT_{em})(1 + pT_2)}{(1 + pT_1)(1 + pT_d)}, \tag{23}$$

where D/D_u is the amplification factor of the inertialess amplifier, T_1, T_d, T_{em}, and T_2 are time constants of the correcting loop. Taking into account (3) and (21) we obtain

$$K_u(p) K_k(p) = K(p) = \frac{D}{p(1 + pT_1)(1 + pT_k)} \frac{1 + pT_2}{(1 + pT_d) \prod_{i=4}^{n} (1 + pT_i)}. \tag{24}$$

After T_k is taken to be equal to T_3, and it is taken into account that T_d and T_1 are small parameters, a comparison of (24) with (22) shows that:

$$K(p) = K_I(p). \tag{25}$$

Thus, the problem is reduced to the determination of the unknown time constants T_1, T_2, and T_d.

For the integro-differential network the following relationships hold true:

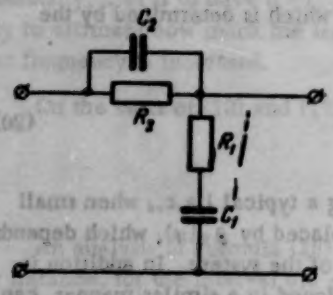

Fig. 3.

$$T_2 T_{em} = T_1 T_d \tag{26}$$

or

$$\frac{\omega_1}{\omega_2} = \frac{\omega_{em}}{\omega_d},$$

where

$$\omega_{em} = \frac{1}{T_{em}},\ \omega_d = \frac{1}{T_d},\ \omega_1 = \frac{1}{T_1} \text{ and } \omega_2 = \frac{1}{T_2}.$$

But because $\frac{\omega_1}{\omega_2} \approx \frac{\omega_c}{D} = \frac{1}{D_c}$ (see appendix) we have:

$$T_d = \frac{\omega_c}{D} T_{em} = \frac{T_{em}}{D_c}. \tag{27}$$

Using the formulas of Table 2 we determine the time constants T_2 and T_1. The sequence of operations for this is as follows: from the given values of φ_c and $\omega_c \sum_{i=4}^{n} T_i$ and the value for T_d just found, we obtain

$$\beta_n(\varphi) = \beta(\varphi) - \frac{1}{2}\omega_c \sum_{i=4}^{n} T_i - \frac{1}{2}\frac{\omega_c^2}{D} T_{em}$$

Then we determine $d = 2 - \frac{\omega_c}{\omega_2}\frac{1}{\beta_n(\varphi)}$ and finally we obtain the sought time constants

$$T_2 = \frac{1}{\omega_2} = \frac{1 - \frac{\omega_c}{D}}{\omega_c d \beta_n(\varphi)},\ T_1 = \frac{1}{\omega_1} = \frac{D}{\omega_c \omega_2}.$$

After T_1 and T_2 have been determined, it is possible to find the parameters of the electrical network from the following relationships:

$$R_2 C_1 \approx T_1 - T_2 - T_{em} \quad R_1 C_1 = T_2, \quad R_2 C_2 = T_{em} \tag{28}$$

For this purpose it is necessary to assume an arbitrary value for one of the sought for parameters, for instance the capacity C_1, and then, using Formulas (28), find the rest of them. In assuming an arbitrary value for the first parameter, a standard size should be chosen, and moreover it should be chosen in such a way that the other parameters required to work with it should correspond or be near to standard sizes listed in a catalog.

In the following a numerical example is given, and to illustrate conditions the open loop logarithmic characteristics of both amplitude and phase versus frequency are plotted * (Fig. 4).

* All numerical examples and also the plotting of the logarithmic characteristics were carried out by L. E. Zalmanova and A. A. Vlasova.

Given $\omega_c = 12\ \text{sec}^{-1}$, $D = 300\ \text{sec}^{-1}$, $T_{em} = 0.1$ sec, $T_k = 0.05$ sec, $T_1 = 0.01$ sec and $\varphi_c = 30°$.

We take T_k to be equal to $T_3 = 0.05$ sec ($\omega_3 = 20\ \text{sec}^{-1}$); then $\beta_n(\varphi) \approx 0.44$, $d = 0.64$, which satisfies inequality (16), $T_2 \approx 0.27$ sec ($\omega_2 = 3.76\ \text{sec}^{-1}$); $T_1 \approx 6.67$ sec ($\omega_1 = 0.15\ \text{sec}^{-1}$).

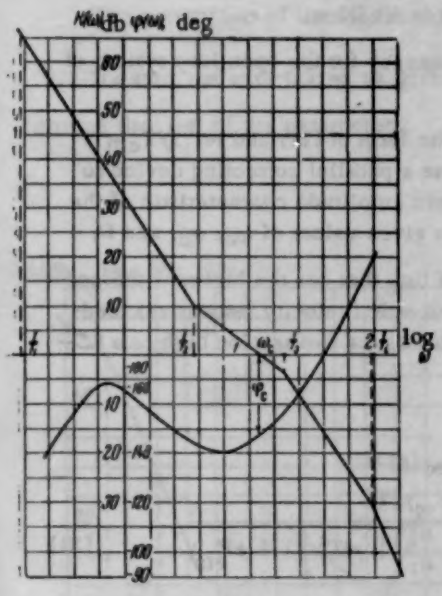

Fig. 4

Now let us determine the parameters of the correcting loop. Let $C_1 = 10\ \mu f$; then $R_1 = T_2/C_1 = 27$ kohm,

$$R_2C_1 \approx 6.3 \text{ sec}, \qquad R_2 = 630 \text{ kohm}.$$

The resistor nearest to 630 kohm in the catalog is 620 kohm. Substituting this value of R_2 into $R_2C_2 = T_{em} = 0.1$ sec, we obtain $C_2 \approx 0.16\ \mu f$. We finally set $C_2 = 0.15\ \mu f$.

2. Suppose that the transfer function for $K_u(p)$ is given in the form of (21), and the time constants T_{em} and T_k are close to each other and both satisfy the inequalities $T_{em} < \frac{1}{\omega_c}$ and $T_k < \frac{1}{\omega_c}$. It is required to choose a correcting device in such a way that the transfer function of the system shall correspond to a l.a.c. of type III, and that ω_c, φ_c, and D attain given values.

On the basis of [2], for a l.a.c. of type III we have

$$K_{III}(p) = \frac{D}{p}\frac{1+pT_3}{(1+pT_1)(1+pT_2)^2}, \tag{29}$$

then it is easy to find, that

$$K_k(p) = \frac{D}{D_u}\frac{1+pT_3}{1+pT_1}, \tag{30}$$

that is, the correcting device represents an inertialess amplifier with an amplification factor of $\frac{D}{D_u}$ and a passive integrating loop, as shown on Fig. 5. K(p) will equal $K_{III}(p)$ if T_3 is made equal to $\sqrt{T_{em}T_k}$, and T_1 is taken into account as a small parameter.

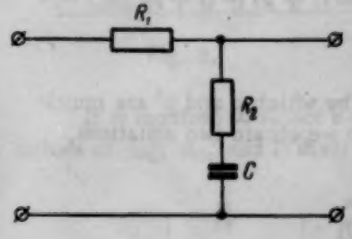

Fig. 5

The problem then is reduced to the determination of T_1 and T_2. The sequence of the procedure is the same as in the first example, except that $\underline{d}$ is found by solving the equation for $\frac{\omega_3}{\omega_c}$ given in Table 2, Column III, which yields:

$$d = 2\left[1 - \frac{\omega_c}{\omega_3\beta_n(\varphi)}\right].$$

The parameters of the integrating loop are determined by means of the formulas:

$$T_1 = \frac{R_1+R_2}{R_2}T_2, \quad T_2 = R_2C.$$

Here is a numerical example. The following data are given: $\omega_c = 5\ \text{sec}^{-1}$, $D = 200\ \text{sec}^{-1}$, $\varphi_c = 30°$, $T_{em} = 0.08$ sec, $T_k = 0.07$ sec, $T_1 = 0.01$ sec. We calculate T_3 to be equal 0.075 sec or $\omega_3 = 13.3\ \text{sec}^{-1}$ and then find:

$$\beta_n(\varphi) \text{ and } d\text{: } \beta_n(\varphi) = \beta(\varphi) - \frac{1}{2}\sum_{i=1}^{n}\omega_i T_i \approx 0.5, \quad d \approx 0.5.$$

Then T_1 and T_2 and also the parameters of the integrating loop are determined:

$$T_2 = \frac{1 - \frac{\omega_c}{D}}{\omega_c d \beta_n(\varphi)} = 0.78 \text{ sec} \quad (\omega_2 = 1.28 \text{ sec}^{-1}), \quad T_1 = \frac{D}{\omega_c \omega_2} = 31.3 \text{ sec} \quad (\omega_1 = 0.032 \text{ sec}^{-1}$$

Let $C = 20\ \mu f$, then $R_2 = \frac{T_2}{C} = 39 \text{ kohm}, R_1 = R_2\left(\frac{T_1}{T_2} - 1\right) \approx 1.6$ megohm.

On Fig. 6 are given log log plots of amplitudes and phases versus frequency for the open loop system of this example.

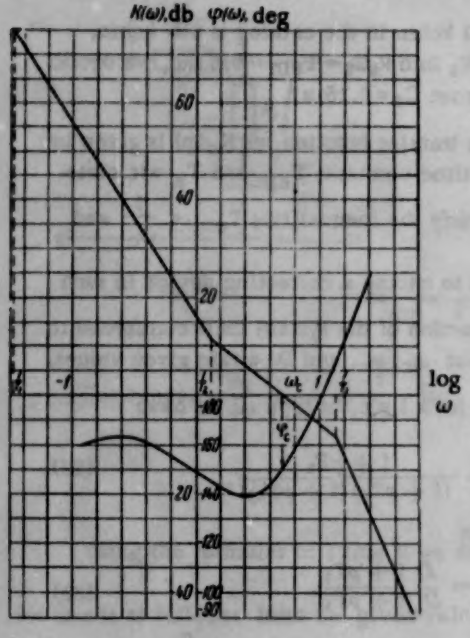

Fig. 6

3. Let $K_u(p)$ have the form of (21) and let $1/T_{em} < \omega_c$. It is sought to choose a parallel correcting device so that the over-all logarithmic amplitude characteristic of the system will be of type II at given values of ω_c, φ_c, and D.

We throw around the link that has the highest time constant T_{em} a flexible tachometric (velocity responsive) feedback line and include in this line a two section high-pass RC-type wave filter (Fig. 7).

Then we have:

$$K_{oc}(p) = \frac{K_{oc} p^2 T_f^2}{\left[1 + pT_f\left(2 + \frac{R_1}{R_2}\right) + p^2 T_f^2\right](1 + pT_{em})} \tag{31}$$

(where T_f is the time constant of the filter), or when $\frac{R_1}{R_2} \ll 2$,

$$K_{oc}(p) \approx \frac{K_{oc} p^2 T_f^2}{(1 + pT_f)^2 (1 + pT_{em})}. \tag{32}$$

Comparison of Equations (4), (21), and (32) shows that the over-all l.a.c. of the system is reduced to type II if we put

$$T_f = T_2, \ T_k = T_3,$$
$$(1 + pT_f)^2 (1 + pT_{em}) + K_{oc} p^2 T_f^2 = (1 + pT_1)^2 (1 + pT_d), \tag{33}$$

where T_d is a small parameter.

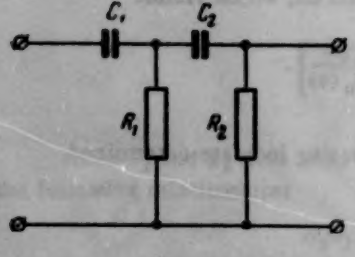

Fig. 7

Equating in (33) the factors by which p and p^2 are multiplied on both sides of the equation we obtain two equations, from which it follows that:

$$T_d = T_{em} \frac{\omega_c}{D},$$
$$K_{oc} \approx \frac{D}{\omega_c} - 1 - \frac{2}{T_2}\left(T_{em} - \sqrt{\frac{D}{\omega_c}}\, T_d\right). \tag{34}$$

Thus, the problem now is reduced to finding T_2 and K_{oc}. The procedure for this is as follows: using the initially prescribed values for ω_c, φ_c, and T_1 and the value for T_d calculated from (34), we find $\beta_n(\varphi)$ by means of the formula:

$$\beta_n(\varphi) = \beta(\varphi) - \frac{1}{2}\omega_c \sum_{i=4}^{n} T_i - \frac{1}{2}\omega_c T_d;$$

Then we determine the value of d from $d = 2 - \frac{\omega_c}{\omega_3 \beta_n(\varphi)}$, where $\omega_3 = \frac{1}{T_k}$, and finally, knowing the numerical values of $\beta_n(\varphi)$ and d, we find:

$$T_2 = \frac{2\left(1 - \sqrt{\frac{\omega_c}{D}}\right)}{\omega_c d \beta_n(\varphi)}$$

and then K_{oc} by means of Formula (34).

The parameters of the RC filter are determined by means of the relationships: $T_f = T_2 = R_2C_2 = R_1C_1$.

We start out with taking as given one of the parameters and the relationship $\frac{R_1}{R_2} \ll 2$ and then determine the rest of the parameters.

We now give a numerical example. The following data are given: $\omega_c = 20\ \text{sec}^{-1}$, $D = 400\ \text{sec}^{-1}$, $T_{em} = 0.1$ sec, $T_k = 0.025$ sec, $T_1 = 0.002$ sec and $\varphi_c = 30°$.

We find $T_d = 0.005$ sec, $d = 0.89$, $\beta_n(\varphi) \approx 0.45$, $T_2 \approx 0.195$ sec, $K_{oc} = 18.2$.

Then for the parameters of the RC filter we have $R_2C_2 = R_1C_1 = 0.195$ sec. Let $C_1 = 2\mu f$ and $R_1/R_2 = 0.1$. Then $R_1 = 97$ kohm. We take $R_1 = 100$ kohm, and we then obtain $R_2 = 1$ meg. and $C_2 = 0.2\ \mu f$.

On Fig. 8 are shown the open loop logarithmic amplitude and phase versus frequency characteristics for this example.

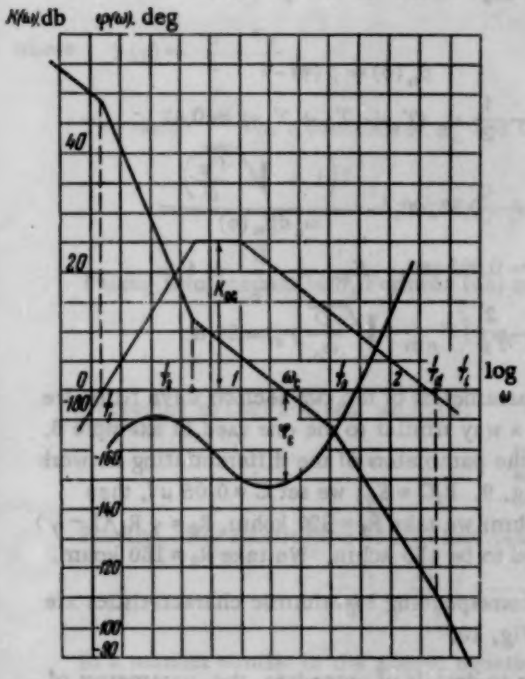

Fig. 8.

4. We are given the open loop transfer function of the uncorrected part of a system.

$$K_u(p) = \frac{D_u}{p} \frac{1}{(1+pT_{em})(1+pT_k)(1+pT_u)(1+pT_y)} \times$$

$$\times \frac{1}{\prod_{i=4}^{n}(1+pT_i)}, \qquad (35)$$

where T_k and T_u are time constants close to each other, both smaller than $\frac{1}{\omega_c}$, and $T_{em} > \frac{1}{\omega_c}$.

It is required to select a correcting arrangement in such a way, that the over-all l.a.c. at the given values of ω_c, φ_c, and D shall be of type IV.

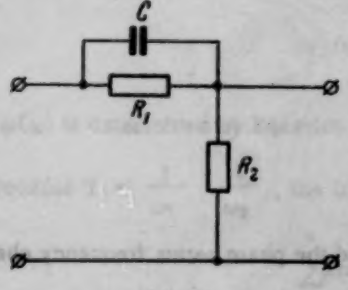

Fig. 9

In solving this problem, we can utilize an inertialess amplifier, a parallel correcting device, which has a transfer function of the form of (32) and a series correcting device which has a transfer function:

$$K_k(p) = \frac{1+pT_y}{1+pT_y\gamma} \quad (\gamma<1), \qquad (36)$$

where γ is chosen from the condition $T_y\gamma < T_2$. This transfer function can be obtained by means of a differentiating electric network (without taking into account the loss) as shown on Fig. 9. The system is then brought to type IV of the l.a.c.'s if we make:

$$T_f = T_2,\ T_2 = \sqrt{T_kT_u},\quad (1+pT_f)^2(1+pT_{em}) + K_{oc}p^2T_f^2 = (1+pT_1)^2(1+pT_d)$$

and account for the time constants T_d and $T_y\gamma$ by considering them as small parameters. Thus, the problem is reduced to finding of T_2 and K_{oc} and to the choice of γ. The values of T_2 and K_{oc} are determined in exactly the same way as in the preceding example; only, the value of $\underline{d}$ must be found by means of the formula:

$$d = 2\left(1 - \frac{\omega_c}{\omega_3 \beta_n(\varphi)}\right).$$

The amplification factor of the inertialess amplifier is chosen to equal $\frac{D}{D_u \gamma}$.

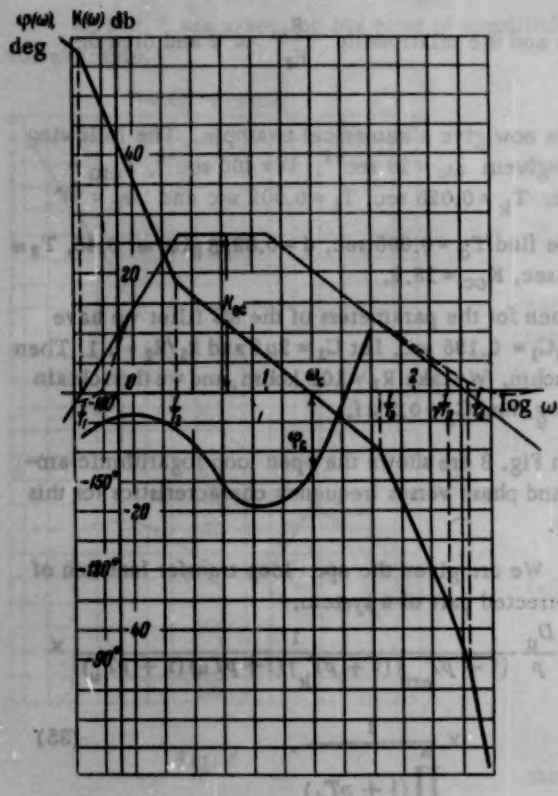

Fig. 10

Let us now introduce a numerical example: Given $\omega_c = 18\ \text{sec}^{-1}$, $D = 400\ \text{sec}^{-1}$, $\varphi_c = 30°$, $T_{em} = 0.1$ sec, $T_k = 0.02$ sec, $T_u = 0.015$ sec, $T_y = 0.03$ sec, $T_i = 0.001$ sec.

We find $T_d = T_{em}\frac{\omega_c}{D} = 0.0045$ sec, $T_3 = \sqrt{T_k T_u} = 0.017$ sec ($\omega_3 = 58\ \text{sec}^{-1}$). Let $\gamma = 0.2$; then

$$\beta_n(\varphi) = \beta(\varphi) - \frac{1}{2}\omega_c (T_i + T_d + T_y\gamma) \approx 0.42,$$

$$d \approx 0.53,\quad T_2 = \frac{2\left(1 - \sqrt{\frac{\omega_c}{D}}\right)}{\omega_c d \beta_n(\varphi)} = 0.395 \text{ sec},\quad K_{oc} = \frac{D}{\omega_c} - 1 - \frac{2}{T_2}\left(T_{em} - \sqrt{\frac{D}{\omega_c}} T_d\right) = 20.8$$

The parameters of the two-section wave filter are selected in a way similar to the one used in example 3. Let us find the parameters of the differentiating network shown on Fig. 9. $R_1 C = T_y$; we set $C = 0.05\ \mu f$, then $R_1 = 600$ kohm; we take $R_1 = 620$ kohm, $R_2 = \gamma R_1/(1-\gamma)$ is calculated to be 155 kohm. We take $R_2 = 150$ kohm.

The corresponding logarithmic characteristics are plotted on Fig. 10.

In a similar manner with respect to the basic idea as well as to details of procedure, the parameters of correcting arrangements can be determined for other uncorrected $K_u(p)$ functions.

APPENDIX

Proof of Equations (5)

It is known that:

$$\varphi_0 = \pi + \varphi(\omega)_{\omega=\omega_c}, \tag{37}$$

$$K(\omega)_{\omega=\omega_c} = 1, \tag{38}$$

where $K(\omega)$ and $\varphi(\omega)$ are respectively the amplitude versus frequency and the phase versus frequency characteristics of an open loop servosystem.

For l.a.c. of type I according to (22), we have:

$$\varphi(\omega) = -\frac{\pi}{2} - \arctan\frac{\omega}{\omega_1} + \arctan\frac{\omega}{\omega_2} - \arctan\frac{\omega}{\omega_3}. \tag{39}$$

For the frequency band $\omega_2 < \omega < \omega_3$ (in the neighborhood of the asymptotic crossover frequency ω_c), the following inequalities hold true:

$$\frac{\omega}{\omega_1}>1, \quad \frac{\omega}{\omega_2}>1, \quad \frac{\omega}{\omega_3}<1. \tag{40}$$

We expand all three arctangent functions $\arctan\frac{\omega}{\omega_i}$ (39) into series, and then use the approximation obtained by discarding all members higher than of the first order. Equation (39) then is changed into:

$$\varphi(\omega)\approx-\frac{\pi}{2}-\frac{\omega_2}{\omega}\left(1-\frac{\omega_1}{\omega_2}\right)-\frac{\omega}{\omega_3}. \tag{41}$$

After substituting (41) at $\omega=\omega_c$ into (37) and rearranging, we obtain:

$$\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_1}{\omega_2}\right)+\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi), \tag{42}$$

where $\beta(\varphi)=\frac{\pi}{4}-\frac{\varphi_c}{2}$.

The ratio $\frac{\omega_1}{\omega_2}$ as a function of ω_c and D is obtained from Equations (22) and (38). From (22)

$$K(\omega)=\frac{D}{\omega}\frac{\sqrt{1+(\omega T_2)^2}}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_3)^2}}. \tag{43}$$

Taking into account (40), Formula (43) can be simplified into

$$K(\omega)\approx\frac{D}{\omega}\frac{\omega_1}{\omega_2}. \tag{44}$$

According to (38) and (44) we have:

$$\frac{\omega_1}{\omega_2}\approx\frac{\omega_c}{D}. \tag{45}$$

Finally, after substitution of (45) into (42) the latter becomes identical with (5):

$$\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_c}{D}\right)+\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi).$$

In a manner similar to the above, equations corresponding to (5) can be obtained for l.a.c.'s of types II, III, and IV.

Accounting For Small Parameters

For transfer Function (19) the phase versus frequency characteristic is:

$$\varphi_1(\omega)=\varphi(\omega)-\sum_{i=4}^{n}\arctan\frac{\omega}{\omega_i}, \tag{46}$$

where $\varphi(\omega)$ is determined by Equation (41).

Because $T_i=\frac{1}{\omega_i}<\frac{1}{\omega_3}$, the inequality $\frac{\omega_c}{\omega_i}\ll 1$ holds true and we have:

$$\sum_{i=4}^{n}\arctan\frac{\omega_c}{\omega_i}\approx\sum_{i=4}^{n}\frac{\omega_c}{\omega_i}=\omega_c\sum_{i=4}^{n}T_i.$$

We then substitute Formula (46) putting $\omega=\omega_c$ into Equation (37) and obtain:

$$\frac{\omega_2}{\omega_c}\left(1-\frac{\omega_c}{D}\right)+\frac{\omega_c}{\omega_3}\approx 2\beta(\varphi)-\omega_c\sum_{i=4}^{n}T_i.$$

Introducing:

$$\beta_n(\varphi) = \beta(\varphi) - \frac{1}{2}\omega_c \sum_{i=1}^{n} T_i,$$

we obtain an equation of the same type as (5) except that on the right side $\beta(\varphi)$ must be replaced by $\beta_n(\varphi)$.

LITERATURE CITED

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AN ANALYTICAL FORMULATION OF THE SYNTHESIS PROBLEM OF CORRECTIVE DEVICES IN LINEAR SERVOSYSTEMS

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An analytical formulation of the generalized problem of synthesizing corrective devices in linear servosystems is proposed, and, on the basis of the initial synthesis equations which are derived, a method is worked out for determining the transfer functions of corrective devices for the most frequently encountered cases of correction. In conclusion, an example is given of a solution of the synthesis problem.

INTRODUCTION

At the present time the least developed branch of the theory of linear electrical circuits with lumped constants and linear automatic control systems is the theory of their synthesis, including the most important of its branches – the theory of synthesizing corrective devices.

Notwithstanding a series of major achievements in this branch which have been reported in [1-6] and in a series of other works, there is lacking up to the present time:

1) An analytical formulation of the synthesis problem for corrective devices;

2) A method of determining the optimal transfer function, which the corrective system's transfer function should be made to equal, and its generalized analytic expression;

3) A formulation of recommendations for the selection of various corrective networks, keeping in mind the peculiarities of the invariant parts of the servosystem and the actual uses to which it is put;

4) A formulation of recommendations as to the expediency of using active corrective devices, again keeping in mind the actual conditions under which they will be used.

In the present paper an attempt is made to formulate the general synthesis problem analytically and generalized transfer function equations for corrective devices are derived subject to the conditions that their expressions take into account requirements of type [7] for the corrective servosystem as well as the peculiarities of its invariant part.

1. Formulation of the General Synthesis Problem of Corrective Devices

Let us take some open linear electric circuit, or open servosystem, or automatic control system made up of N directed sections, where the transfer function for the i-th section is known to be

$$W_i(p) = \frac{A_{0,i} + A_{1,i}p + A_{2,i}p^2 + \ldots + A_{s_i,i}p^{s_i}}{p^{j_i}(B_{0,i} + B_{1,i}p + B_{2,i}p^2 + \ldots + B_{q_i,i}p^{q_i})}. \qquad (1)$$

The entire open circuit's transfer function is

$$K(p) = \prod_{i=1}^{N} W_i(p) = K_0 \frac{C_0 + C_1 p + C_2 p^2 + \ldots + C_m p^m}{p^r (D_0 + D_1 p + D_2 p^2 + \ldots + D_n p^n)}. \quad (2)$$

Let us assume that this servosystem is not functioning properly and that m special corrective quadripoles (CQP) are inserted into the corrective circuit (C-circuit) to correct this.

It is also assumed that the transfer function (2) corresponds to the invariant part of the servosystem and that the only variable parameter in this servosystem is the resulting constant gain factor. Consequently, it is also possible to vary the constant gain factor of separate sections.*

On the basis of the foregoing, the transfer function of the C-circuit after correction is

$$K_G(p) = K_{0G} \frac{(C_0 \pm \Delta C_0) + (C_1 \pm \Delta C_1) p + \ldots + (C_m \pm \Delta C_m) p^m}{p^{r \pm \Delta F} [(D_0 \pm \Delta D_0) + (D_1 \pm \Delta D_1) p + \ldots + (D_n \pm \Delta D_n) p^n]} \frac{S(p)}{U(p)} = \quad (3)$$

$$= \prod_{i=1}^{N} W_i^*(p) \frac{Sp}{Up},$$

which we will call the desired transfer function (GF) because it is the result of the corrections.

In Expression (3), we designate

$$K_{cG} = (K_0 \pm \Delta K_0) \frac{(C_0 \pm \Delta C_0)}{(D_0 \pm \Delta D_0)}$$

as the new constant gain factor of the servosystem, and $\pm \Delta K_0$, $\pm \Delta C_0$, $\pm \Delta D_0$, $\pm \Delta F$, $\pm \Delta C_i$ as the increments added to the corresponding constant coefficients of the C-circuit's transfer function as a result of the correction, while S(p) and U(p) are polynomials in p:

$$S(p) = 1 + S_1 p + \ldots + S_\lambda p^\lambda,$$
$$U(p) = 1 + U_1 p + \ldots + U_\gamma p^\gamma.$$

It is assumed that since the synthesis problem has been solved, it is possible to determine the set of CQPs, introduced into the C-circuit, the plan for their insertion, and the analytic expression for the transfer function of each of the CQPs.

It is assumed further that the corrective networks selected, and the set of CQPs inserted, make it possible to get the simplest analytic expressions for their transfer functions.

It is assumed further that:

1) The introduction into the circuit of active corrective devices (CQPA) is undesirable because of their relative complexity as compared with passive corrective devices (CQPP) and can be justified only in those cases where the use of CQPP does not yield the desired results;

2) Since the method of determining the circuit of a CQP, given its transfer function, is known [8-10] and will not be considered in the present paper, the solution of the synthesis problem in the present paper will lead to the determination of the corrective device's transfer function. As far as imposing realizability requirements on the expressions we have calculated for these transfer functions is concerned, this matter will be dealt with in another article.

* Although this limitation complicates the proposed problem, it also makes it possible to solve the synthesis problem for such systems, the invariant part of which consists of standard assemblies and components, where in most cases the only variable parameter is the constant gain factor.

2. Networks For Introducing Corrective Devices

To get exhaustive recommendations as to the expediency of using various corrective networks, it is necessary, obviously, to consider the greatest possible number of variants of these networks.

In the author's opinion, the networks for introducing CQPs into C-circuits which are listed below and are depicted in general form in Figs. 1-9, include the overwhelming majority of corrective networks known to the literature. As far as the nomenclature for all the varieties of corrective networks that are listed is concerned, it is quite arbitrary and is introduced solely for convenience of presentation.

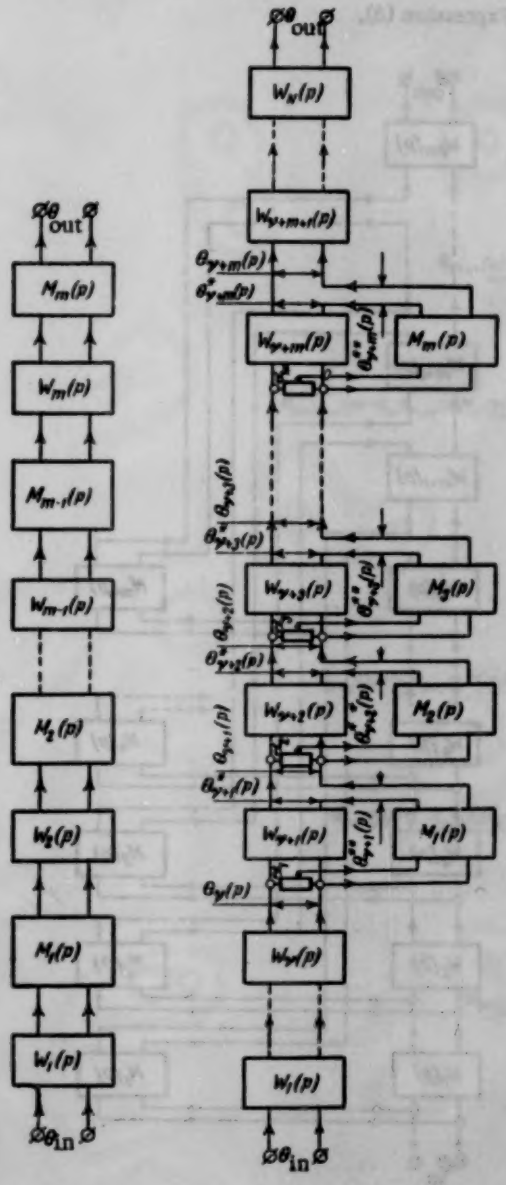

Fig. 1

Fig. 2

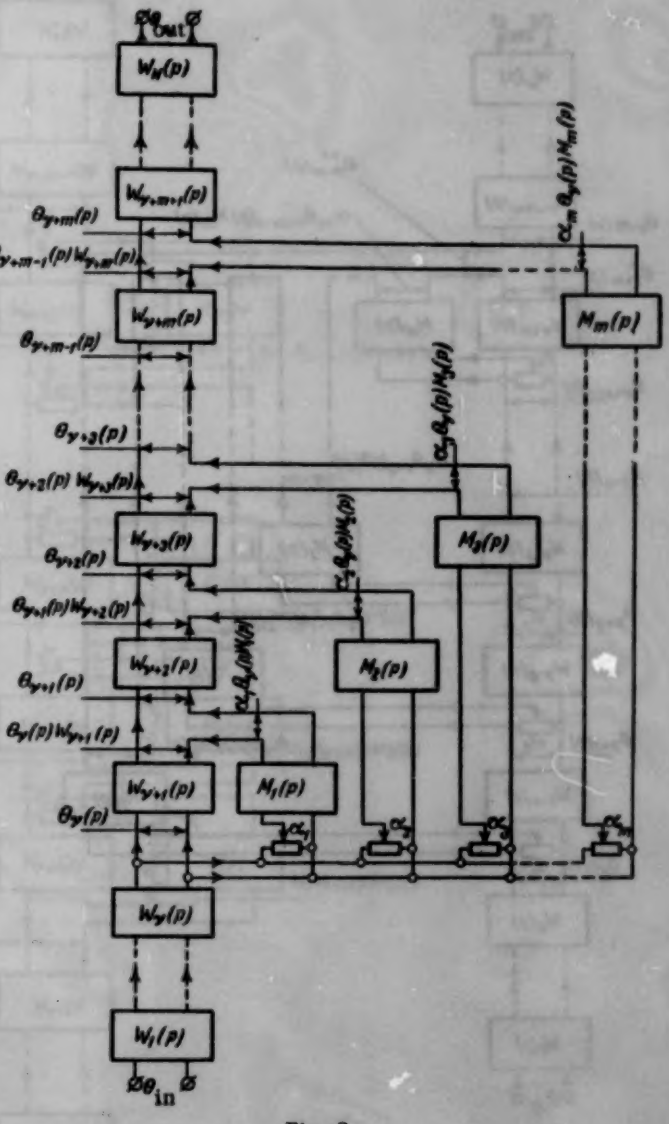

Fig. 3

Appendix I gives solutions of systems of operator equations which correspond to all the corrective networks depicted in Figs. 1-9. The analytic expression for $K_1(p)$ - $K_9(p)$ which is calculated there is the transfer function for the C-circuit, modified by the insertion of the indicated network of corrective devices. In these expressions the unknown transfer functions for the CQPs which have been introduced appear as the operator functions $M_i(p)$ which we try to determine, where i ($i = 1, \ldots, m$) corresponds to the number of the corrective device.

3. The Generalized Equations of Synthesis of Corrective Devices and Their Solution

In accord with the above the most general form of the synthesis equation is

$$K_\sigma(p) = K_G(p), \tag{4}$$

where $K_\sigma(p)$ is the transfer function of the corrective circuit after the insertion of $\underline{m}$ corrective devices according to any of the networks depicted in Figs. 1-9 (see the corresponding expression for $K_1(p)$-$K_9(p)$ in Appendix I), σ is the number ($\sigma = 1, \ldots, 9$) of the corrective network corresponding to Figs. 1-9, $K_G(p)$ is the desired transfer function, in general form corresponding to Expression (3).

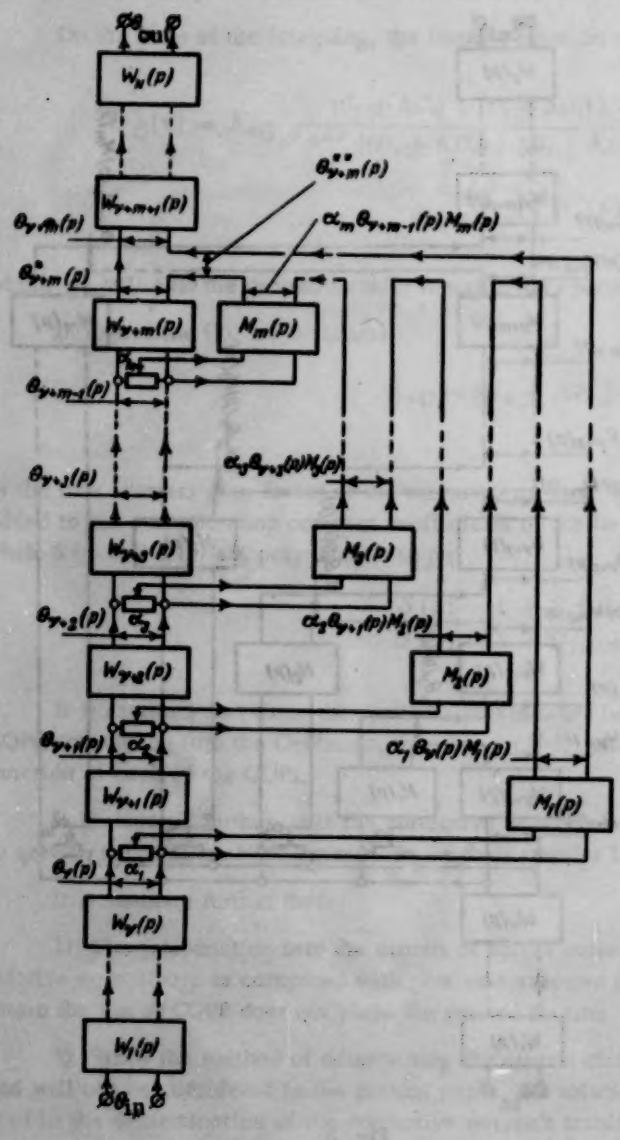

Fig. 4

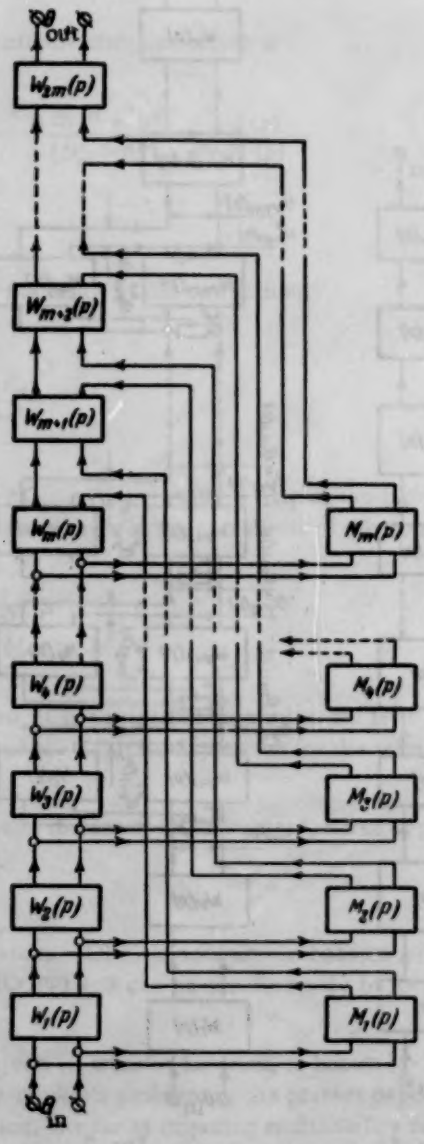

Fig. 5

We will recall that in Equation (4) and its particular cases, the transfer functions $M_i(p)$ are the expressions sought.

Particular cases of the synthesis Equation (4) are considered in detail in Appendix I, corresponding to all nine types of corrective networks (Equations I-IX) considered in the present paper.

The solution in general form of the above-mentioned equations makes it possible to determine the analytic expressions for the corrective devices' transfer functions which in this case will have the following form.

A. For series correction networks

$$\prod_{i-1}^{m} M_i(p) = \prod_{i-1}^{N-m} \frac{W_i^*(p)}{W_i(p)} \frac{S(p)}{U(p)} . \qquad (5)$$

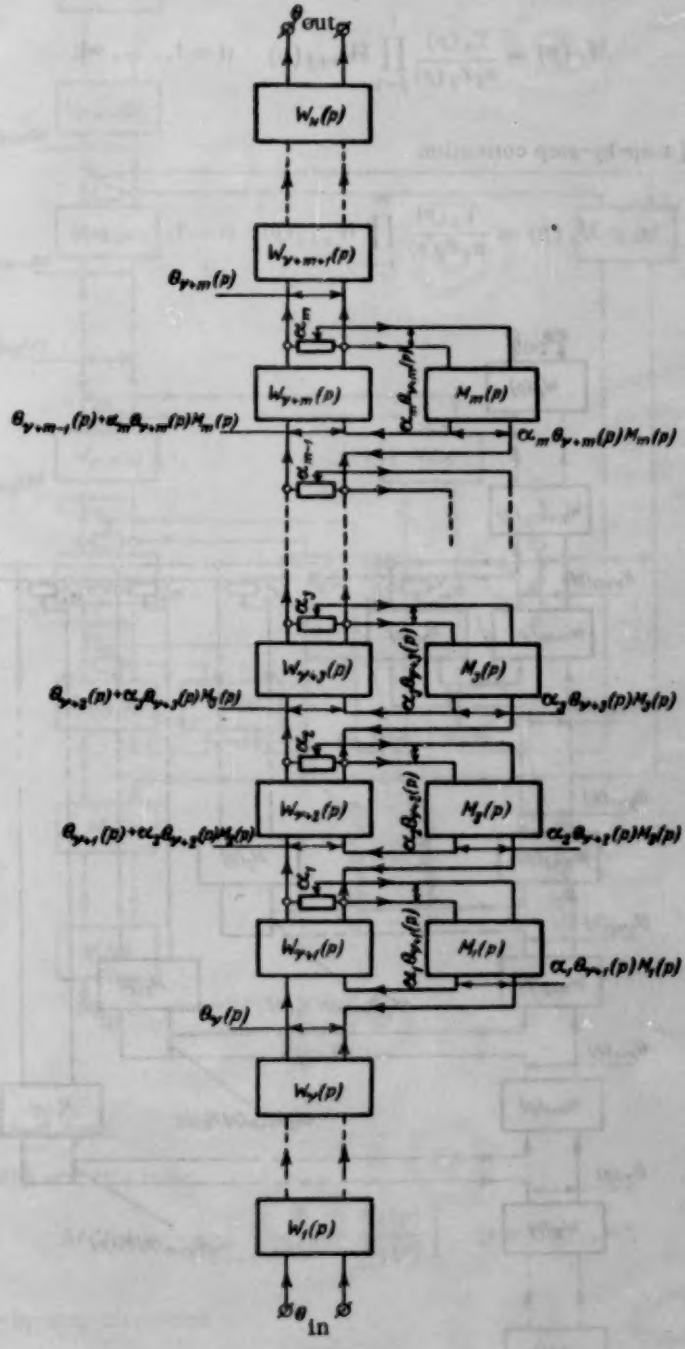

Fig. 6

B. For parallel correction networks

I. Cases of the directly-connected type:

1) Method of bypassing one at a time

$$M_i(p) = \frac{1}{\alpha_i} W_{\nu+i}(p)\left[\frac{Y_i(p)}{Z_i(p)} - 1\right] \quad (i = 1, \ldots, m); \tag{6}$$

2) First type of step-by-step correction

$$M_i(p) = \frac{Y_i(p)}{\alpha_i Z_i(p)} \prod_{j=1}^{i} W_{\nu+j}(p) \quad (i = 1, \ldots, m); \tag{7}$$

3) Second type of step-by-step correction

$$M_i(p) = \frac{Y_i(p)}{\alpha_i Z_i(p)} \prod_{j=i}^{m} W_{\nu+j}(p) \quad (i = 1, \ldots, m); \tag{8}$$

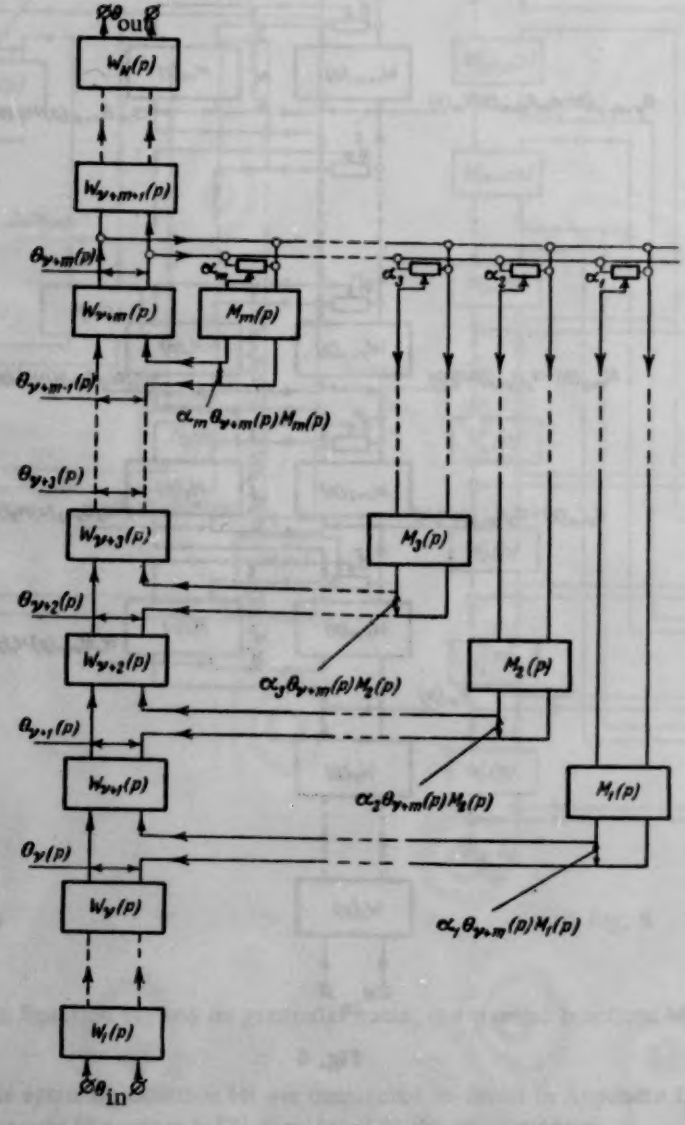

Fig. 7

4) Method of cross-switching

$$M_i(p) = \frac{Y_i(p)}{\alpha_i Z_i(p)} \prod_{j=i}^{m} W_j(p) \prod_{j=1}^{i-1} W_{m+j}(p) \quad (i = 1, \ldots, m). \tag{9}$$

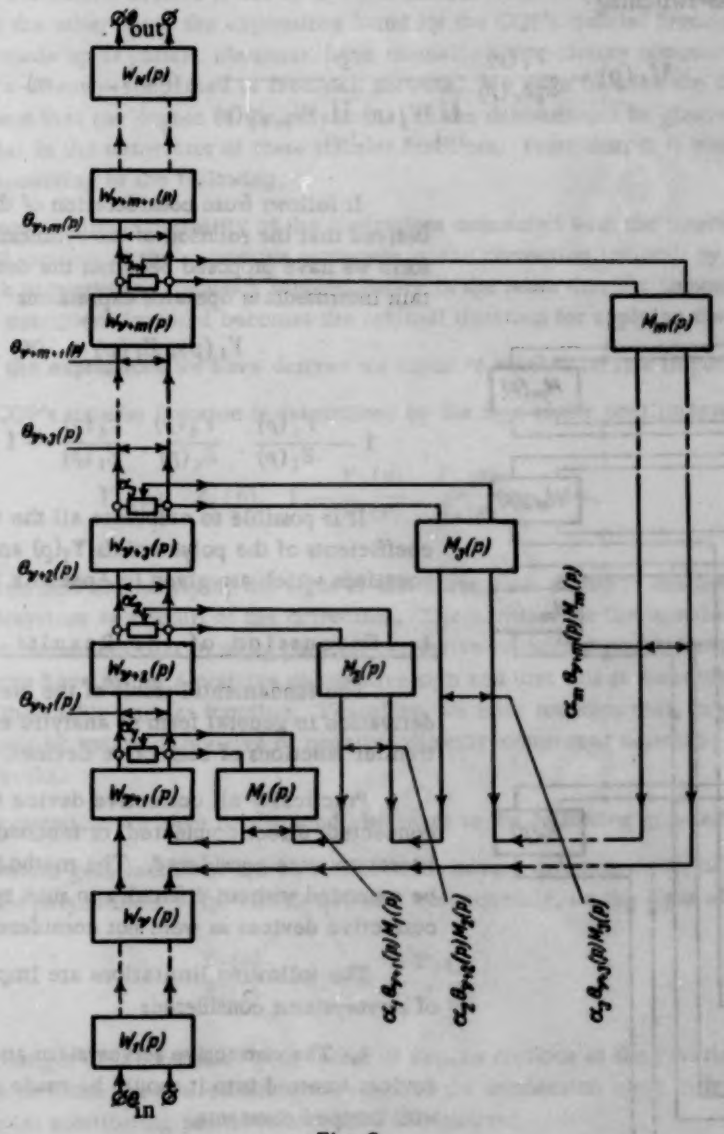

Fig. 8

II. Cases of the feedback type:

1) Method of bypassing one at a time

$$M_i(p) = \frac{1}{\alpha_i W_{v+i}(p)} \left[1 - \frac{Y_i(p)}{Z_i(p)}\right] \quad (i = 1, \ldots, m); \tag{10}$$

2) First type of step-by-step correction

$$M_i(p) = \frac{Y_i(p)}{\alpha_i Z_i(p)} \frac{1}{\prod_{j=i}^{m} W_{v+j}(p)} \quad (i = 1, \ldots, m); \tag{11}$$

3) Second type of step-by-step correction

$$M_i(p) = \frac{Y_i(p)}{\alpha_i Z_i(p)} \frac{1}{\prod_{j=1}^{i} W_{\nu+j}(p)} \quad (i = 1, \ldots, m); \tag{12}$$

4) Method of cross-switching

$$M_i(p) = \frac{Y_i(p)}{\alpha_i Z_i(p)} \frac{1}{\prod_{j=i}^{m} W_j(p) \prod_{j=1}^{i-1} W_{m+j}(p)} \quad (i = 1, \ldots, m). \tag{13}$$

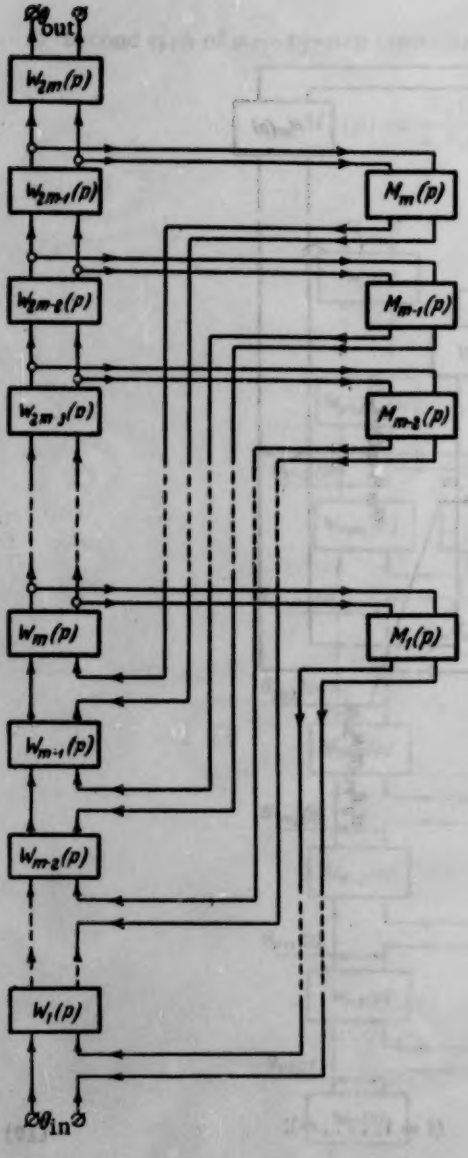

Fig. 9

It follows from consideration of the equations we have derived that the solution of the synthesis problem in the form we have proposed becomes the determination of certain intermediate operator expressions

$$Y_i(p),\ Z_i(p) \quad \text{or}$$

$$1 - \frac{Y_i(p)}{Z_i(p)},\ \frac{Y_i(p)}{Z_i(p)},\ \frac{Y_i(p)}{Z_i(p)} - 1 \quad \text{etc.}$$

It is possible to calculate all the unknown constant coefficients of the polynomials $Y_i(p)$ and $Z_i(p)$ from equations which are given in Appendix I.

4. Discussion of the Results

The fundamental result of the present paper is the derivation in general form of analytic expressions for the transfer functions of corrective devices.

Practically all corrective device networks of the series connected, direct connected, or feedback types known to the literature were considered. The methods developed also may be extended without difficulty to such types of insertion of corrective devices as were not considered here.

The following limitations are imposed on the class of servosystems considered:

1. The corrective servosystem and the corrective devices inserted into it should be made up of linear sections with lumped constants.

2. The sections inserted into the servosystem should be detecting (directed action).

As we see, the limitations listed are minimal. They do not prevent consideration of a wide class of the most frequently encountered linear automatic control systems and servosystems, made up of circuits with lumped constants, and linear electrical circuits in which wave phenomena may be neglected, assuming, that they were made up entirely of lumped constants.

Passing over to a discussion of the problem of actually constructing equipment corresponding to the corrective devices' transfer functions, it is necessary to note the following.

All the expressions we have gotten for the transfer functions discussed may be constructed as active or passive CQPs or combinations of both, depending on the correction problem involved.

Analyzing the expressions we have gotten, we may also note that imposing the requirements listed on these transfer functions in its turn makes necessary certain restrictions on the structure of the transfer functions which make corrections in various sections of the C-circuit. In this way, requiring passive CQPs or imposing other conditions on the corrective devices is not always compatible with the peculiarities of the invariant part of the servosystem. On the other hand, the expressions found for the CQP's transfer functions, when these corrective devices are made up of passive elements, have mutually contradictory structures when these corrective devices are inserted as a directly-connected or feedback network. We have in mind the differences involved in satisfying the requirement that the degree of the polynomial in the denominator be greater or equal to the degree of the polynomial in the numerator of these transfer functions. From this, it is possible to make a far-reaching conclusion, amounting to the following.

By virtue of the mutual incompatibility of the limitations connected with the structural peculiarities of the transfer functions of sections of the corrective servosystem, the correction methods by means of directly-connected and feedback networks are mutually supplementary in the sense that the impossibility of using one method because of the conditions imposed becomes the optimal situation for applying the other method.

Upon analysis of the expressions we have derived we come to another no less important conclusion.

The sign of the CQP's transfer function is determined by the sign of the coefficients in the intermediate expressions

$$Y_i(p),\ Z_i(p),\ 1-\frac{Y_i(p)}{Z_i(p)},\ \frac{Y_i(p)}{Z_i(p)}-1 \text{ etc.}$$

that is, the sign is determined ultimately by the signs of the increments, added to the time constants in the invariant part of the servosystem as a result of the correction. Therefore, when the question is put in general terms, it is necessary to remember that in using parallel corrective networks,* each insertion of a CQP into the corrective circuit can have either a positive or negative sign and that this is determined by the sign of the analytic expression of the CQP's transfer function. Therefore, we must mention that, in general, CQP's may be inserted in a C-circuit by means of positive or negative directly-connected networks, or by positive or negative feedback networks.

Analysis of the expressions we have determined also leads to the following generalizations.

1. Since the constant gain factors of the various sections have a direct relation to the absolute magnitude of the coefficients of the polynomials $Y_i(p)$ and $Z_i(p)$, and consequently, on the signs of their differences in the expressions

$$\frac{Y_i(p)}{Z_i(p)}-1,\quad 1-\frac{Y_i(p)}{Z_i(p)}$$

etc., then by suitable changes of the constant gain factor of various sections of the invariant part of the servosystem** it is possible to make the desired changes in sign of the connection when different corrective devices are inserted (for example, substituting positive feedback for negative).

2. Only through a suitable and therefore unique choice as to how to connect the corrective network (directly-connected or feedback) does the proposed synthesis problem become solvable when it is required that the analytic expressions of the corrective devices' transfer functions, given a certain invariant part of the servosystem and a certain desired transfer function, correspond to the use of passive corrective devices. However, the use of active corrective devices in solving a practical synthesis problem may permit the partial, and in some cases, the complete relaxation of the structural limitations which were established here, and which must be applied to the transfer functions of sections of the servosystem's invariant part or to the corrective devices' transfer functions. Nevertheless, because of the mutual incompatibility of the limitations associated with the

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* In one given servosystem.

** See footnote on p. 138.

structural peculiarities of sections of the servosystem's invariant part, when using directly-connected or feedback type networks, use of CQPA is usually inexpedient and can be justified only in those rare cases where the corrective network is subject to severe restrictions and the correction problem can not be solved using only CQPP.

In the present paper, the problem proposed was to find general analytic expressions for CQP's transfer functions for the broadest class of networks that are inserted into C-circuits, subject to the condition that GF also be given in general form.

The next step in the investigation is the establishment of all those requirements which must be satisfied by the servosystem's invariant part and by the CQP's transfer functions in order that the synthesis problem can be solved in the simplest manner.

APPENDIX I

1. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 1. The initial system of equations is

$$\theta_1(p) = W_1(p)\,\theta_{in}(p),$$

$$\theta_1^*(p) = \theta_1(p)\,M_1(p),$$

$$\theta_2(p) = \theta_1^*(p)\,W_2(p),$$

$$\theta_2^*(p) = \theta_2(p)\,M_2(p),$$

$$\cdots\cdots\cdots$$

$$\theta_m(p) = \theta_{m-1}^*(p)\,W_m(p),$$

$$\theta_{out}(p) = \theta_m^*(p) = \theta_m(p)\,M_m(p).$$

The unknown transfer function is the solution of the following system of equations:

$$K_1(p) = \frac{\theta_{out}(p)}{\theta_{in}(p)} = \prod_{i=1}^{N-m} W_i(p) \prod_{i=1}^{m} M_i(p).$$

The synthesis equation is derived by substituting in (4) the expressions $K_1(p)$ and $K_G(p)$ and getting

$$\prod_{i=1}^{N-m} W_i(p) \prod_{i=1}^{m} M_i(p) = \prod_{i=1}^{N} W_i^*(p) = \frac{S(p)}{U(p)}. \qquad (I)$$

The solution of this equation with respect to the m unknown corrective devices' transfer functions gives Expression (5) (see text of article).

2. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 2.

The initial system of equations is:*

$$\theta_{\nu+1}^*(p) = \theta_\nu(p)\,W_{\nu+1}(p),$$

$$\theta_{\nu+1}^{**}(p) = \alpha_1\theta_\nu(p)\,M_1(p)$$

$$\theta_{\nu+1}(p) = \theta_{\nu+1}^*(p) + \theta_{\nu+1}^{**}(p) = [W_{\nu+1}(p) + \alpha_1 M_1(p)]\,\theta_\nu(p),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\theta_{\nu+m}^*(p) = \theta_{\nu+m-1}(p)\,W_{\nu+m}(p),$$

* Here, and below, with the exception of the cross-switching cases for the CQPs, it is assumed that the corrective devices are inserted in the C-circuit somewhere between the $\nu+1$ and $\nu+m$ sections, and that the coefficient α_i gives the fraction of the signal that is found in the given part of the C-circuit.

$$\theta^{**}_{\nu+m}(p) = \alpha_m \theta_{\nu+m-1}(p) M_m(p),$$

$$\theta_{\nu+m}(p) = \theta^{*}_{\nu+m}(p) + \theta^{**}_{\nu+m}(p) = [W_{\nu+m}(p) + \alpha_m M_m(p)]\, \theta_{\nu+m-1}(p),$$

$$\theta_{\nu}(p) = \theta_{in}(p) \prod_{i=1}^{\nu} W_i(p),$$

$$\theta_{out}(p) = \theta_{\nu+m}(p) \prod_{i=\nu+q+1}^{N} W_i(p).$$

The unknown transfer function is the solution of this system of equations

$$K_2(p) = \prod_{i=1}^{N} W_i(p) \prod_{i=1}^{m} \left[1 + \frac{\alpha_i M_i(p)}{W_{\nu+i}(p)}\right].$$

By substituting $K_2(p)$ and $K_G(p)$ in (4) we get the synthesis equation as

$$\prod_{i=1}^{N} W_i(p) \prod_{i=1}^{m} \left[1 + \frac{\alpha_i M_i(p)}{W_{\nu+i}(p)}\right] = \prod_{i=1}^{N} W^{*}_i(p) \frac{S(p)}{U(p)}. \qquad \text{(II)}$$

Expression (6) for the i-th corrective section may be found by solving the above equation. To find the unknown coefficients of the polynomials $Y_i(p)$, $Z_i(p)$ in Expression (6) we consider the equation

$$\prod_{i=1}^{m} \frac{Y_i(p)}{Z_i(p)} = \prod_{i=1}^{N} \frac{W^{*}_i(p)}{W_i(p)} \frac{S(p)}{U(p)},$$

which in expanded form is

$$\prod_{i=1}^{m} Y_i(p) = \prod_{i=1}^{N} \mathfrak{Y}^{*}_i(p) \prod_{i=1}^{N} \mathfrak{Z}_i(p)\, S(p),$$

$$\prod_{i=1}^{m} Z_i(p) = \prod_{i=1}^{N} \mathfrak{Y}_i(p) \prod_{i=1}^{N} \mathfrak{Z}^{*}_i(p)\, U(p),$$

where we assume

$$W_i(p) = \frac{\mathfrak{Y}_i(p)}{\mathfrak{Z}_i(p)}, \quad W^{*}_i(p) = \frac{\mathfrak{Y}^{*}_i(p)}{\mathfrak{Z}^{*}_i(p)},$$

$\mathfrak{Y}_i(p)$, $\mathfrak{Y}^{*}_i(p)$, $\mathfrak{Z}_i(p)$, $\mathfrak{Z}^{*}_i(p)$ are polynomials in p.

3. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 3.

The initial system of equations is

$$\theta^{*}_{\nu+1}(p) = \theta_{\nu}(p) W_{\nu+1}(p),$$

$$\theta^{**}_{\nu+1}(p) = \alpha_1 \theta_{\nu}(p) M_1(p),$$

$$\theta_{\nu+1}(p) = \theta^{*}_{\nu+1}(p) + \theta^{**}_{\nu+1}(p) = [W_{\nu+1}(p) + \alpha_1 M_1(p)]\, \theta_{\nu}(p),$$

. . . . . .

$$\theta^{*}_{\nu+m}(p) = \theta_{\nu+m-1}(p) W_{\nu+m}(p),$$

$$\theta^{**}_{\nu+m}(p) = \alpha_m \theta_\nu(p) M_m(p),$$

$$\theta_{\nu+m}(p) = \theta^{*}_{\nu+m}(p) + \theta^{**}_{\nu+m}(p) = W_{\nu+m}(p)\theta_{\nu+m-1}(p) + \alpha_m M_m(p)\theta_\nu(p),$$

$$\theta_\nu(p) = \theta_{in}(p)\prod_{i=1}^{\nu} W_i(p),$$

$$\theta_{out}(p) = \theta_{\nu+m}(p)\prod_{i=\nu+m+1}^{N} W_i(p).$$

The unknown transfer function is found by solving the following system of equations:

$$K_3(p) = \prod_{i=1}^{N} W_i(p)\left[1 + \sum_{i=1}^{m} \frac{\alpha_i M_i(p)}{\prod_{j=1}^{i} W_{\nu+j}(p)}\right].$$

By analogy with the foregoing the synthesis equation is

$$\prod_{i=1}^{N} W_i(p)\left[1 + \sum_{i=1}^{m} \frac{\alpha_i M_i(p)}{\prod_{j=1}^{i} W_{\nu+j}(p)}\right] = \prod_{i=1}^{N} W^{*}_i(p)\frac{S(p)}{U(p)}. \qquad \text{(III)}$$

The solution of this equation gives Expression (7) for the i-th corrective section. The unknown coefficients of its polynomials $Y_i(p)$, $Z_i(p)$ may be found from the equation

$$\sum_{i=1}^{m} \frac{Y_i(p)}{Z_i(p)} = \prod_{i=1}^{N} \frac{W^{*}_i(p)}{W_i(p)} \frac{S(p)}{U(p)} - 1,$$

which may be written in expanded form as

$$\sum_{i=1}^{m}\prod_{j=1}^{i-1} Z_j(p)\,Y_i(p)\prod_{j=i+1}^{m} Z_j(p) = \prod_{i=1}^{N} \mathrm{Ч}^{*}_i(p)\prod_{i=1}^{N} З_i(p)\,S(p) - \prod_{i=1}^{N} \mathrm{Ч}_i(p)\prod_{i=1}^{N} З^{*}_i(p)\,U(p),$$

$$\prod_{i=1}^{m} Z_i(p) = \prod_{i=1}^{N} \mathrm{Ч}_i(q)\prod_{i=1}^{N} З^{*}_i(p)\,U(p).$$

4. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 4.

The initial system of equations is

$$\theta_{\nu+m}(p) = \theta^{*}_{\nu+m}(p) + \theta^{**}_{\nu+m}(p),$$

$$\theta^{*}_{\nu+m}(p) = \theta_\nu(p)\prod_{i=1}^{m} W_{\nu+i}(p),$$

$$\theta^{**}_{\nu+m}(p) = \alpha_1 M_1\theta_\nu(p) + \cdots + \alpha_{\nu+m}M_{\nu+m}(p)\theta_{\nu+m-1}(p) =$$

$$= \sum_{i=1}^{m}\alpha_i M_i(p)\theta_{\nu+i-1}(p),$$

$$\theta_{\nu+i-1}(p) = \theta_\nu(p)\prod_{j=1}^{i-1} W_{\nu+j}(p),$$

$$\prod_{j=1}^{i-1} W_{\nu+j}(p) = 1, \text{ if } \quad i-1 < 1.$$

$$\theta_v(p) = \theta_{in}(p) \prod_{i=1}^{v} W_i(p),$$

$$\theta_{out}(p) = \theta_{v+m}(p) \prod_{i=v+m+1}^{N} W_i(p).$$

The unknown transfer function is found by solving the following system of equations:

$$K_4(p) = \prod_{i=1}^{N} W_i(p) \left[1 + \sum_{i=1}^{m} \frac{\alpha_i M_i(p)}{\prod_{j=i}^{m} W_{v+j}(p)} \right].$$

By analogy with the foregoing the synthesis equation is

$$\prod_{i=1}^{N} W_i(p) \left[1 + \sum_{i=1}^{m} \frac{\alpha_i M_i(p)}{\prod_{j=i}^{m} W_{v+j}(p)} \right] = \prod_{i=1}^{N} W_i^*(p) \frac{S(p)}{U(p)}. \qquad \text{(IV)}$$

The solution of this equation gives Expression (8) for the i-th corrective section. The equations for finding the unknown coefficients of the polynomials $Y_i(p)$, $Z_i(p)$ are analogous in general form to the previous case.

5. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 5.

The initial system of equations is

$$\theta_m^*(p) = \theta_{in}(p) \prod_{i=1}^{m} W_i(p),$$
$$\theta_m^{**}(p) = \theta_{in}(p) M_1(p),$$
$$\theta_m(p) = \theta_m^*(p) + \theta_m^{**}(p),$$
$$\cdots\cdots\cdots\cdots$$
$$\theta_{2m-1}^*(p) = \theta_{2m-2}(p) W_{2m-1}(p),$$
$$\theta_{2m-1}^{**}(p) = \theta_{m-1}(p) M_m(p),$$
$$\theta_{2m-1}(p) = \theta_{2m-1}^*(p) + \theta_{2m-1}^{**}(p),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_{2m}(p) = \theta_{out}(p) = \theta_{2m-1}(p) W_{2m}(p).$$

The unknown transfer function is found by solving the following system of equations:

$$K_5(p) = \prod_{i=1}^{2m-N} W_i(p) \left[1 + \sum_{i=1}^{m} \frac{\alpha_i M_i(p)}{\prod_{j=i}^{m} W_j(p) \prod_{j=1}^{i-1} W_{m+j}(p)} \right].$$

By analogy with the foregoing the synthesis equation is

$$\Pi W_i(p) \left[1 + \sum_{i=1}^{m} \frac{\alpha_i M_i(p)}{\prod_{j=i}^{m} W_j(p) \prod_{j=1}^{i-1} W_{m+j}(p)} \right] = \prod_{i=1}^{2m-N} W_i^*(p) \frac{S(p)}{U(p)}. \qquad \text{(V)}$$

The solution of this equation gives Expression (9) for the i-th corrective section. The equations for finding the unknown coefficients of the polynomials $Y_i(p)$, $Z_i(p)$ are analogous in general form to the previous two cases.

6. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 6.

The initial system of equations is

$$\theta_{\nu+1}(p) = [\theta_{\nu}(p) + \alpha_1 M_1(p)\,\theta_{\nu+1}(p)]\,W_{\nu+1}(p),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_{\nu+i}(p) = [\theta_{\nu+i-1}(p) + \alpha_i M_i(p)\,\theta_{\nu+i}(p)]\,W_{\nu+i}(p),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_{\nu+m}(p) = [\theta_{\nu+m-1}(p) + \alpha_m M_m(p)\,\theta_{\nu+m}(p)]\,W_{\nu+m}(p),$$
$$\theta_{\nu}(p) = \theta_{in}(p)\prod_{i=1}^{\nu} W_i(p),$$
$$\theta_{out}(p) = \theta_{\nu+m}(p)\prod_{i=\nu+m+1}^{N} W_i(p).$$

The unknown transfer function is found by solving the following system of equations:

$$K_s(p) = \prod_{i=1}^{N} W_i(p)\prod_{i=1}^{m}\frac{1}{1-\alpha_i M_i(p)\,W_{\nu+i}(p)}.$$

By analogy with the foregoing the synthesis equation is

$$\prod_{i=1}^{N} W_i(p)\prod_{i=1}^{m}\frac{1}{1-\alpha_i M_i(p)\,W_{\nu+i}(p)} = \prod_{i=1}^{N} W_i^*(p)\,\frac{S(p)}{U(p)}. \qquad \text{(VI)}$$

The solution of this equation gives Expression (10) for the i-th corrective section. To determine the unknown coefficients of the polynomials $Y_i(p)$, $Z_i(p)$ in (10) we consider the equation

$$\prod_{i=1}^{m}\frac{Z_i(p)}{Y_i(p)} = \prod_{i=1}^{N}\frac{W_i^*(p)}{W_i(p)}\,\frac{S(p)}{U(p)},$$

which may be written in expanded form as

$$\prod_{i=1}^{m} Z_i(p) = \prod_{i=1}^{N} \mathfrak{q}_i^*(p)\prod_{i=1}^{N} \mathfrak{Z}_i(p)\,S(p),$$
$$\prod_{i=1}^{m} Y_i(p) = \prod_{i=1}^{N} \mathfrak{q}_i(p)\prod_{i=1}^{N} \mathfrak{Z}_i^*(p)\,U(p).$$

7. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 7.

The initial system of equations is

$$\theta_{\nu+m}(p) = [\theta_{\nu+m-1}(p) + \alpha_m M_m(p)\,\theta_{\nu+m}(p)]\,W_{\nu+m}(p),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_{\nu+m-i}(p) = [\theta_{\nu+m-i+1}(p) + \alpha_{m-i} M_{m-i}(p)\,\theta_{\nu+m}(p)]\,W_{\nu+m-i}(p),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_{\nu+1}(p) = [\theta_{\nu}(p) + \alpha_1 M_1(p)\,\theta_{\nu+m}(p)]\,W_{\nu+1}(p),$$
$$\theta_{\nu}(p) = \theta_{in}(p)\prod_{i=1}^{\nu} W_i(p),$$

$$\theta_{out}(p) = \theta_{v+m}(p) \prod_{i=v+m+1}^{N} W_i(p).$$

The unknown transfer function is found by solving the following system of equations:

$$K_7(p) = \prod_{i=1}^{N} W_i(p) \frac{1}{1 - \sum_{i=1}^{m} \alpha_i M_i(p) \prod_{j=i}^{m} W_{v+j}(p)}.$$

By analogy with the foregoing the synthesis equation is

$$\prod_{i=1}^{N} W_i(p) \frac{1}{1 - \sum_{i=1}^{m} \alpha_i M_i(p) \prod_{j=i}^{m} W_{v+j}(p)} = \prod_{i=1}^{N} W_i^*(p) \frac{S(p)}{U(p)}. \qquad \text{(VII)}$$

The solution of this equation gives Expression (11) for the i-th corrective section. To determine the unknown coefficients of the polynomials $Y_i(p)$, $Z_i(p)$ in (11) we consider the equation

$$\sum_{i=1}^{m} \frac{Y_i(p)}{Z_i(p)} = 1 - \prod_{i=1}^{N} \frac{W_i(p)}{W_i^*(p)} \frac{U(p)}{S(p)},$$

which may be written in expanded form as

$$\sum_{i=1}^{m} \prod_{j=1}^{i-1} Z_j(p)\, Y_i(p) \prod_{j=i+1}^{m} Z_j(p) =$$

$$= \prod_{i=1}^{N} \mathfrak{Y}_i^*(p) \prod_{i=1}^{N} \mathfrak{Z}_i(p)\, S(p) - \prod_{i=1}^{N} \mathfrak{Y}_i(p) \prod_{i=1}^{N} \mathfrak{Z}_i^*(p)\, U(p),$$

$$\prod_{i=1}^{m} Z_i(p) = \prod_{i=1}^{N} \mathfrak{Y}_i^*(p) \prod_{i=1}^{N} \mathfrak{Z}_i(p)\, S(p).$$

8. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 8.

The initial system of equations is

$$\theta_v^*(p) = \theta_{in}(p) \prod_{i=1}^{v} W_i(p),$$

$$\theta_v^{**}(p) = \sum_{i=1}^{m} \alpha_i M_i(p)\, \theta_{v+i}(p),$$

$$\theta_v(p) = \theta_v^*(p) + \theta_v^{**}(p),$$

. . . . . .

$$\theta_{v+i}(p) = \frac{\theta_{v+m}(p)}{\prod_{j=i+1}^{m} W_{v+j}(p)} \quad (i = 1, \ldots, m-1).$$

. . . . . .

$$\theta_{v+m}(p) = \theta_v(p) \prod_{i=1}^{m} W_{v+i}(p),$$

$$\theta_{out}(p) = \theta_{v+m}(p) \prod_{i=v+m+1}^{N} W_i(p).$$

The unknown transfer function is found by solving the following system of equations:

$$K_3(p) = \prod_{i=1}^{N} W_i(p) \frac{1}{1 - \sum_{i=1}^{m} \alpha_i M_i(p) \prod_{j=1}^{i} W_{v+j}(p)}.$$

By analogy with the foregoing the synthesis equation is

$$\prod_{i=1}^{N} W_i(p) \frac{1}{1 - \sum_{i=1}^{m} \alpha_i M_i(p) \prod_{j=1}^{i} W_{v+j}(p)} = \prod W_i^*(p) \frac{S(p)}{U(p)}. \qquad \text{(VIII)}$$

The solution of this equation gives Expression (12) for the i-th corrective section. The equations for finding the unknown coefficients of the polynomials $Y_1(p)$, $Z_1(p)$ are analogous in general form to the previous case.

9. Find the C-circuit's transfer function when corrective devices are inserted as in the network in Fig. 9.

The initial system of equations is

$$\begin{aligned}
\theta_{in}^*(p) &= \theta_m(p) M_1(p),\\
\theta_{in}(p) &= \theta_{in}(p) + \theta_{in}^*(p),\\
&\cdots\cdots\cdots\cdots\\
\theta_{m-1}(p) &= \theta_{m-2}^*(p) W_{m-1}(p),\\
\theta_{m-1}^*(p) &= \theta_{2m-1}(p) M_m(p),\\
\theta_{m-1}^*(p) &= \theta_{m-1}(p) + \theta_{m-1}^*(p),\\
\theta_m(p) &= \theta_{m-1}^*(p) W_m(p),\\
&\cdots\cdots\cdots\cdots\\
\theta_{2m}(p) &= \theta_{out}(p) = \theta_m(p) \prod_{i=m+1}^{2m} W_i(p).
\end{aligned}$$

The unknown transfer function is found by solving the following system of equations:

$$K_9(p) = \prod_{i=1}^{2m-N} W_i(p) \frac{1}{1 - \sum_{i=1}^{m} \alpha_i M_i(p) \prod_{j=1}^{m} W_j(p) \prod_{j=1}^{i-1} W_{m+j}(p)}.$$

By analogy with the foregoing the synthesis equation is

$$\prod_{-1}^{N-2m} W_i(p) \frac{1}{1 - \sum_{i=1}^{m} \alpha_i M_i(p) \prod_{j=i}^{m} W_j(p) \prod^{i-1} W_{m+j}(p)} = \prod_{i=1}^{N-2m} W_i^*(p) \frac{S(p)}{U(p)}. \qquad \text{(IX)}$$

The solution of this equation gives Expression (13) for the i-th corrective section. The equations for finding the unknown coefficients of the polynomials $Y_1(p)$, $Z_1(p)$ are analogous in general form to the previous two cases.

APPENDIX II

An example of a solution of the synthesis problem

Given a servosystem with first-order instability whose transfer function is

$$K(p) = \frac{K_0}{p} W_1(p) W_2(p) W_3(p) W_4(p) W_5(p),$$

where

$$K_0 = 100 \frac{1}{\text{sec}},$$

$$W_1(p) = K_1 \frac{(1+U_1p)(1+U_2p)}{(1+T_1p)(1+T_2p)} = \frac{(1+6.25p)(1+0.667p)}{(1+7.8p)(1+0.535p)},$$

$$W_2(p) = K_2 \frac{(1+U_3p)(1+U_4p)}{(1+T_3p)(1+T_4p)} = 8 \frac{(1+0.71p)(1+0.00667p)}{(1+0.952p)(1+0.005p)},$$

$$W_3(p) = \frac{1}{(1+T_5p)} = \frac{1}{(1+1.56p)},$$

$$W_4(p) = \frac{1}{(1+T_6p)} = \frac{1}{(1+0.00125p)}, \quad W_5(p) = \frac{1}{(1+T_7p)} = \frac{1}{(1+0.0009p)}.$$

The correction problem is reduced to the following.

It is necessary to find: numerical formulas for the transfer functions of the corrective devices which will be used in feedback networks; their number; where they will be inserted in the servosystem, etc. As a result of the correction, the servosystem should satisfy the following requirements: 1) have first order instability; 2) after the completion of one step of step-by-step control it should not have an overcorrection σ exceeding 25%; 3) it should not have a transient time exceeding 0.85 seconds after the completion of one step of step-by-step control. Finally, the constant gain factor of the entire open servosystem and its various branches should remain unchanged after the correction.

Using the methods described in [1], it is possible to calculate the optimal transfer function, which the servosystem should have after correction:

$$K_{\text{opt}}(p) = \frac{K_0(1+U_1p)(1+U_2p)(1+U_3p)(1+U_4p)}{p[1+p(T_1+\Delta T_1)][1+(T_2-\Delta T_2)p][1+(T_3+\Delta T_3)p][1+(T_4-\Delta T_4)p]} \times$$

$$\times \frac{1}{(1+T_5p)(1+T_6p)(1+T_7p)} = \frac{100(1+6.25p)(1+0.667p)}{p(1+10p)(1+0.003p)(1+p)} \times$$

$$\times \frac{(1+0,71p)(1+0,00667p)}{(1+0.004p)(1+1.56p)(1+0.00125p)(1+0.0009p)},$$

where $\Delta T_1 = 2.2$ seconds, $\Delta T_2 = -0.532$ seconds, $\Delta T_3 = 0.048$ seconds, $\Delta T_4 = -0.001$ seconds.

Turning to the choice of corrective device circuit diagrams, we note that the following conditions are implied by Expressions (10) - (13). In order to be certain of an exact solution of the synthesis problem while using passive corrective devices (not modifying the optimal transfer function by inserting additional inertial sections), it is necessary to include in the correction scheme only those sections whose transfer functions have a numerator whose order is not less than the denominator's order.

To satisfy these correction requirements only the first and second sections can be used. In order to get the simplest network of corrective devices, we will insert them into the corrective servosystem by the method of bypassing one at a time.*

From the above, and using (10) and (VI), we get

* A detailed comparison of the correction circuits considered in this paper will be made in another article.

$$\frac{Y_1}{Z_1}\frac{Y_2}{Z_2}=\frac{[1+(T_1+\Delta T_1)\,p]\,[1+(T_2-\Delta T_2)\,p]\,[1+(T_3+\Delta T_3)\,p]\,[1+(T_4-\Delta T_4)\,p]}{(1+T_1p)(1+T_2p)(1+T_3p)(1+T_4p)}=$$

$$=\frac{(1+10p)(1+0.003p)(1+p)(1+0.004p)}{(1+7.8p)(1+0.535p)(1+0.952p)(1+0.005p)}$$

and further

$$Z_1(p)=1+Z_{1,1}p+Z_{1,2}p^2=(1+7.8p)(1+0.535p),$$
$$Z_2(p)=1+Z_{1,2}p+Z_{2,2}p^2=(1+0.952p)(1+0.005p).$$

In contrast to the way the polynomials $Z_i(p)$ were chosen, it is necessary to define the polynomials $Y_i(p)$ so that the difference $Z_i(p)-Y_i(p)$ should have coefficients all of the same sign. Only one case satisfies this requirement:

$$Y_1(p)=1+Y_{1,1}p+Y_{1,2}p^2=(1+10p)(1+p),$$
$$Y_2(p)=1+Y_{2,1}p+Y_{2,2}p^2=(1+0.0033p)(1+0.004p),$$

since in this case the difference of the polynomials

$$Z_1(p)-Y_1(p)=(Z_{1,1}-Y_{1,1})\,p+(Z_{1,2}-Y_{1,2})\,p^2=-(2.66p+5.83p^2),$$
$$Z_2(p)-Y_2(p)=(Z_{2,1}-Y_{2,1})\,p+(Z_{2,2}-Y_{2,1})\,p^2=0.95p+0.00475p^2$$

does not have coefficients of different sign.

Using this method to calculate the unknown coefficients, and Formula (10), which is relevant here, we compute the unknown formulas for the corrective devices' transfer functions:

$$M_1(p)=\frac{1}{W_1(p)}\left[1-\frac{Y_1(p)}{Z_1(p)}\right]=-2.66\,\frac{p(1+2.19p)}{(1+6.25p)(1+0.667p)},$$
$$M_2(p)=\frac{1}{W_2(p)}\left[1-\frac{Y_2(p)}{Z_2(p)}\right]=0.119\,\frac{p(1+0.005p)}{(1+0.71p)(1+0.0067p)}.$$

Since the method of proceeding from the expressions we have derived to actual electrical networks has been worked out in sufficient detail, we consider that the solution of the proposed synthesis problem has been completed.

We confirm the correctness of the result by substituting the expressions we have computed into the formula for $K_\phi(p)$. Since $K_\phi(p)$ is found to equal the optimal transfer function, we conclude that the synthesis problem was solved correctly.

In conclusion, we note that having computed the corrective devices' transfer functions, the signs of these functions indicate that the first should be inserted into the servosystem as a negative feedback network, the second as a positive feedback network.

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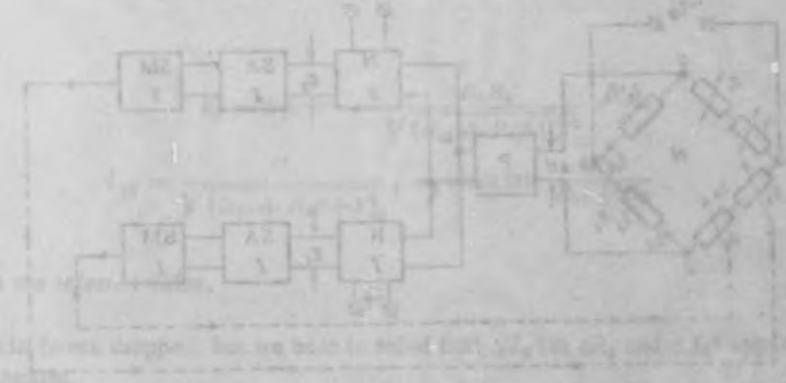

AN AC BRIDGE WITH SELF-BALANCING IN TWO PARAMETERS

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An ac bridge with self-balancing in two parameters in the same arm is considered. The effects of interconnection between the self-balancing circuits and of nonlinearity in the static characteristics of the bridge on the stability are analyzed. A bridge circuit in which it is possible to prevent interaction between the self-balancing circuits over a wide range of measured quantities is presented.

1. Introduction

Two time-varying components of an impedance (R and X, C and tan δ, etc.) have often to be measured accurately in automatic process controls.

An ac bridge with self-balancing in two parameters is the best device for the purpose. The circuits for this so far given can be divided into two classes, according to the method of producing the signals which control the two driven elements.

In the first group, described in [1-3], the control signals are produced by splitting the output voltage from the bridge diagonal into two components using two phase-sensitive null detectors. Figure 1 shows the block diagram of one such bridge. In general both control signals (ϵ_1 and ϵ_2) are functions of both controlled parameters, and hence the balancing circuits are coupled.

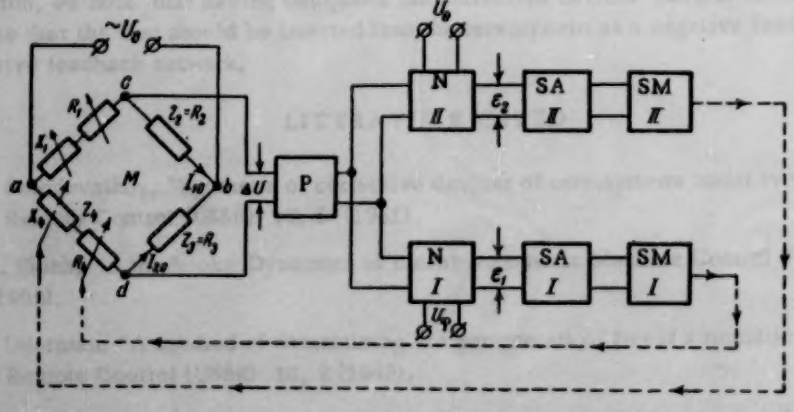

Fig. 1. Block diagram of a bridge. M) bridge circuit; P) preamplifier; N) phase-sensitive null detectors; SA) servoamplifiers; SM) servomotors; U_φ and U_θ) reference voltages of phase angles φ and θ relative to I_{10}.

In the second group system proposed by Karandeev et al. [3, 4] the control signals are produced by using two special differential indicators (differential balancing detectors). Each detector is supplied via phase-shifting networks with two voltages taken from different points in the bridge. The detectors produce signals which mainly depend on one parameter only, which means that it is possible to prevent interaction between the self-balancing circuits over a wide range of measured quantities.

However, these detectors cause a considerable increase in the static error (up to 2%) [5]. Karandeev and Grinevich [6] have effected some reductions in this source of error. None the less, it is very much more difficult to design these detectors for stability and high precision than in the case of the first group of circuits.

The first-group bridges have only been analyzed [2-5] in relation to the measurement side. It has therefore been (incorrectly) supposed [3-5] that the coupling between the balancing circuits and the nonlinearity of the static characteristics will mean that the first group will in principle have unsatisfactory transient response.

The above argument shows that the first group is of interest, and that an analysis of the transient response is of value. The method of producing the control signals using the phase-sensitive detectors is equally applicable to complex automatic balancing devices, which cannot be said about the use of differential null indicators.

Here we consider the stability of one type of group-one circuit, namely a four-arm bridge with two identical servodrives which actuate the resistive and reactive components in one arm (Fig. 1).

Krasovskii's complex coordinates method [7] can be used in the analysis.

We deal here with the effect of the static nonlinearity and of the balancing circuit coupling on the stability.

2. The Stability of a Linearized System

In drawing up the equations of motion for the circuit of Fig. 1 we neglect the transients in the bridge proper (see Appendix) and use the measurement diagonal voltage in complex form:

$$\dot{U} = \dot{U}_0 \left(\frac{\dot{Z}_3}{\dot{Z}_3 + \dot{Z}_4} - \frac{\dot{Z}_2}{\dot{Z}_1 + \dot{Z}_2} \right). \tag{1}$$

At small deviations from balance the static characteristics are arcs of circles and can be replaced by their tangents at the balance point. Let subscript 0 denote values at balance; then

$$\Delta\dot{U} = \left(\frac{\partial \dot{U}}{\partial \dot{Z}_4} \right)_0 \Delta\dot{Z}_4 + \left(\frac{\partial \dot{U}}{\partial \dot{Z}_1} \right)_0 \Delta\dot{Z}_1 = -\dot{k}_1 (\Delta\dot{Z}'_4 - \Delta\dot{Z}_1), \tag{2}$$

where

$$\dot{k}_1 = k_1 e^{-j\alpha_0} = \frac{I_{10} R_2}{\sqrt{(R_{10} + R_2)^2 + X_{10}^2}} e^{-j\alpha_0}$$

$$I_{10} = \frac{U_0}{\sqrt{(R_{10} + R_2)^2 + X_{10}^2}}, \quad \alpha_0 = \arctan \frac{X_{10}}{R_{10} + R_2}. \tag{3}$$

$\Delta\dot{Z}'_4 = \Delta\dot{Z}_4 \frac{R_2}{R_3}$ is the referred value.

The primes are in future dropped, but we bear in mind that $\Delta\dot{Z}_4$ (or ΔR_4 and ΔX_4) appearing later are actually referred quantities.

If the reference voltage of phase-sensitive detector I differs in phase by an angle φ from the current in branch one ($\dot{I}_{10}$), and that of II by an angle $\theta = \varphi + \frac{\pi}{2}$, the output voltages ϵ_1 and ϵ_2 from the detectors when these operate as switches are, respectively,

$$\varepsilon_1 = k_2 \mathrm{Re}\,[\Delta\dot{U}e^{j\varphi}] = -k_1k_2\mathrm{Re}\,[e^{j(\varphi-\alpha_0)}(\Delta\dot{Z}_4 - \Delta\dot{Z}_1)],$$
$$\varepsilon_2 = k_2 \mathrm{Re}[\Delta\dot{U}e^{j\left(\varphi-\frac{\pi}{2}\right)}] = k_2\,\mathrm{Im}\,[\Delta\dot{U}e^{j\varphi}] = -k_1k_2\,\mathrm{Im}\,[e^{j(\varphi-\alpha_0)}(\Delta\dot{Z}_4 - \Delta\dot{Z}_1)], \quad (4)$$

where k_2 is the gain of P and of the detectors.

Since $\Delta\dot{Z}_1 = \Delta R_1 + j\Delta X_1$ and $\Delta\dot{Z}_4 = \Delta R_4 + j\Delta X_4$ (4) becomes

$$\varepsilon_1 = -k_1k_2\,[\cos(\varphi-\alpha_0)(\Delta R_4 - \Delta R_1) - \sin(\varphi-\alpha_0)(\Delta X_4 - \Delta X_1)],$$
$$\varepsilon_2 = -k_1k_2\,[\sin(\varphi-\alpha_0)(\Delta R_4 - \Delta R_1) + \cos(\varphi-\alpha_0)(\Delta X_4 - \Delta X_1)]. \quad (4a)$$

Adding to (4) or (4a) the equation for the servomotors plus amplifiers and balancing elements

$$(Tp+1)\,p\Delta R_4 = k_3k_4\varepsilon_1, \quad (Tp+1)\,p\Delta X_4 = k_3k_4\varepsilon_2, \quad (5)$$

where $p = \frac{d}{dt}$, k_3 is the amplifier gain, k_4 is the gain of servomotor plus reducing gear and balancing element and T the servomotor time-constant, we get a system of equations describing the action in the linear approximation. The voltage amplifiers and detectors can in most cases be taken as lagless.

Figure 2 shows the equivalent structural diagram of an automatic bridge compiled from (4a) and (5), which shows that the bridge is a two-channel servo with direct antisymmetric dc feedbacks with relation to the input parameters (ΔR_1 and ΔX_1) and to the controlled parameters (ΔR_4 and ΔX_4). A special feature is that the coupling constants $k_c = \sin(\varphi - \alpha_0)$ can be altered via the phases of the reference voltages.

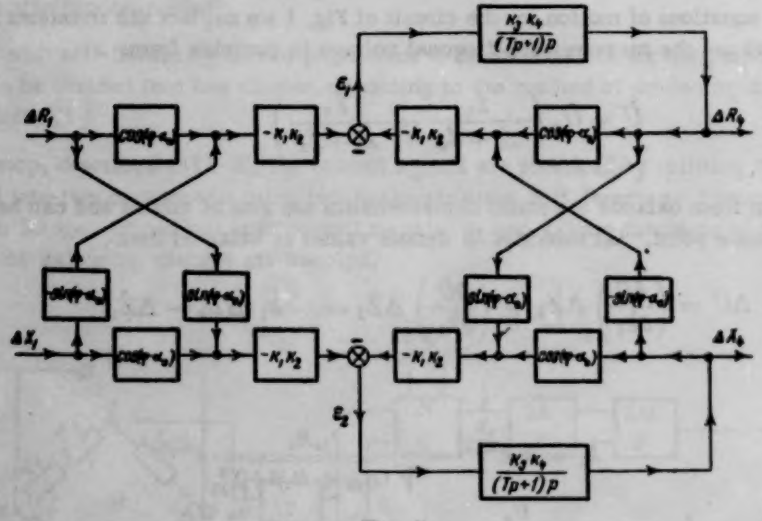

Fig. 2

We now examine how the coupling constants affect the properties of the system. To produce the necessary sensitivity in measuring the input parameters we must have a gain given by

$$K = k_1k_2k_3\cos(\varphi-\alpha_0) = \frac{U_s}{\delta}, \quad (6)$$

where U_s is the servomotor starting voltage and δ the minimum change in the input to which the system must react.

Equation (6) shows that as the coupling constants (i.e., $|\varphi - \alpha_0|$) increase, $k_1k_2k_3$ must be increased to keep K constant.

Hence the open-loop gain $K_p = k_1k_2k_3k_4\cos(\varphi - \alpha_0)$ must be independent of $|\varphi - \alpha_0|$.

The stability condition implies that we must have

$$K_p < K_{p.cr}, \tag{7}$$

where $K_{p.cr}$ is the critical open-loop gain.

We use the complex coordinates method [7] to determine how $K_{p.cr}$ depends on the coupling constants.

Combining (4) and (5), and applying the Laplace transform (initial conditions zero) we have

$$(TS+1)\,S\Delta R_4(S) = -k_s \operatorname{Re}\{[\Delta\dot{Z}_4(S) - \Delta\dot{Z}_1(S)]\,e^{j(\varphi-\alpha_0)}\},$$
$$(TS+1)\,S\Delta X_4(S) = -k_s \operatorname{Im}\{[\Delta\dot{Z}_4(S) - \Delta\dot{Z}_1(S)]\,e^{j(\varphi-\alpha_0)}\}. \tag{8}$$

where $k_s = k_1k_2k_3k_4$, and $\Delta R_4(S)$, $\Delta X_4(S)$, $\Delta\dot{Z}_4(S)$, and $\Delta\dot{Z}_1(S)$ are the Laplace transforms of ΔR_4, ΔX_4, $\Delta\dot{Z}_4$, and $\Delta\dot{Z}_1$. Multiplying the second equation in (8) by j and adding to the first, and using the fact that $\Delta R_4(S) + j\Delta X_4(S) = \Delta\dot{Z}_4(S)$, we get the equation for the system in complex form:

$$[(TS+1)S + k_s e^{j(\varphi-\alpha_0)}]\,\Delta\dot{Z}_4(S) = k_s e^{j(\varphi-\alpha_0)}\Delta\dot{Z}_1(S). \tag{9}$$

The closed-loop transfer function is

$$\Phi_3(S) = \frac{\Delta\dot{Z}_4(S)}{\Delta\dot{Z}_1(S)} = \frac{W_p(S)}{1+W_p(S)}, \tag{10}$$

where

$$W_p(S) = \frac{k_s e^{j(\varphi-\alpha_0)}}{(TS+1)S} \tag{11}$$

is the open-loop transfer function.

For stability $1 + W_p(S)$ must have no zero in the right half-plane of S, and so $W_p(j\omega)$ must not encompass the point $[-1, j0]$ when ω varies between $-\infty$ and $+\infty$.

Equation (11) implies that the circuit coupling amounts to rotating the function $W_c(j\omega) = \dfrac{k_s}{(Tj\omega+1)\,j\omega}$ (the complex transfer function for the separate balancing circuits, with $\varphi = \alpha_0$) through an angle $(\varphi - \alpha_0)$ determined by the degree of coupling.

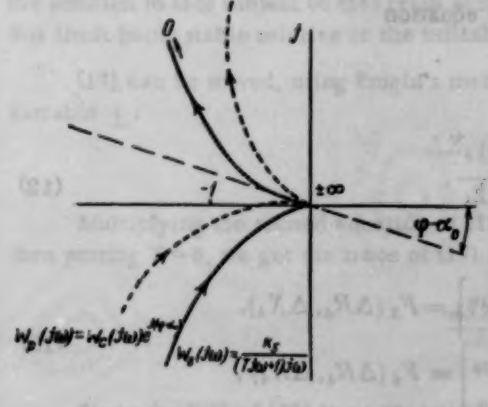

Fig. 3

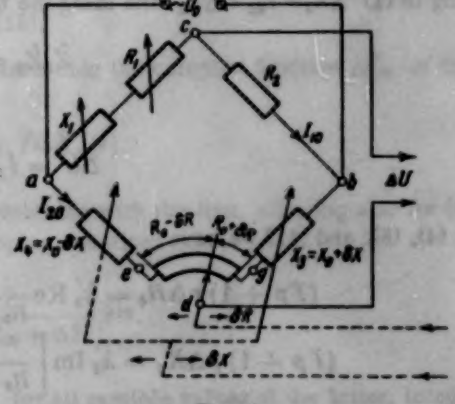

Fig. 4

For this $W_c(j\omega)$, and also for other types found in automatic bridges, rotation about the origin means that the hodograph of $W_c(j\omega)$ approaches the critical point $[-1, j0]$, i.e., that $K_{p.cr}$ is reduced (Fig. 3).

Hence $K_{p.cr}$ decreases as k_c increases, while to keep the sensitivity and response time (which depend on k_4) constant, K_p must be kept constant.

This leads directly to the conclusion that the balancing circuits should be uncoupled.

As the circuit of Fig. 1 has nonlinear static characteristics the balancing circuits can only be uncoupled (at small deviations) by appropriate reference voltage phase selection at one point in the working range. At other points α will differ from $\alpha_0 = \varphi$.

The point where $\varphi = \alpha_0$ is selected within the working range such that the coupling at all other points is the least possible.

We must point out here that these bridges can be designed to have balancing circuits uncoupled over the entire working range. This can occur if we connect the balancing elements as in Fig. 4 [8], i.e., in such a way that the current in the arm remains fixed. Here the static characteristics (balancing lines) are straight, as the topographical diagram of the adb branch (Fig. 5) shows; uncoupling at one point implies uncoupling at all.

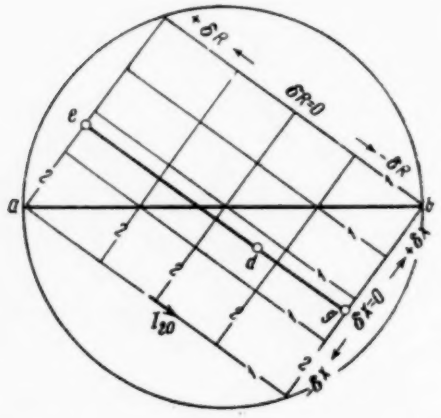

Fig. 5. Topographical diagram for the adb branch. 1) Balance line on adjusting the resistive arm; δR = var, δX = const; 2) balance line on adjusting the reactive arm, δX = var, δR = const. The points a, e, d, g, u, and b in the diagram correspond to the points of a, e, d, g, u, and b in the bridge.

An additional advantage of the circuit of Fig. 4 over that of Fig. 1 is that capacitative and inductive reactances can both be measured without switching, since the potential point corresponding to d can move into either the upper or lower half-planes in Fig. 5 as the balancing elements are adjusted.

A fault often found in the circuit of Fig. 4 is that the readings are not separate, and are frequency-dependent. However, these features are not essential in certain applications. An example is the checking of dimensions and conductivity of parts using eddy currents with an inductive transducer [9]. Here the checking method is frequency-dependent. The frequency must be known, and the two measured parameters, e.g. diameter and conductivity, are both related to the resistive and reactive components of the transducer impedance, so the two parameters cannot be read off separately on a normal bridge either.

3. Stability Allowing For Static Characteristic Nonlinearity

Consider the behavior when ΔZ_4 is large but does not fall outside the range in which coordinate bounding can be neglected.

Putting in (1) $\Delta \dot{Z}_4 = \dot{Z}_{40} + \Delta \dot{Z}_4$ and using the balanced bridge equation

$$\dot{Z}_1 \dot{Z}_3 = \dot{Z}_{40} \dot{Z}_2,$$

we get

$$\Delta \dot{U} = \dot{I}_{10} \frac{-\dot{Z}_2 \Delta \dot{Z}_4}{\dot{Z}_3 + \dot{Z}_{40} + \Delta \dot{Z}_4}. \tag{12}$$

From (4), (5), and (12) we have

$$\begin{aligned} (Tp+1)\, p\Delta R_4 &= k_5' \operatorname{Re}\left[\frac{\Delta \dot{Z}_4}{R_3 + \dot{Z}_{40} + \Delta \dot{Z}_4}\, e^{j\varphi}\right] = F_3(\Delta R_4, \Delta X_4), \\ (Tp+1)\, p\Delta X_4 &= k_5' \operatorname{Im}\left[\frac{\Delta \dot{Z}_4}{R_3 + \dot{Z}_{40} + \Delta \dot{Z}_4}\, e^{j\varphi}\right] = F_4(\Delta R_4, \Delta X_4), \end{aligned} \tag{13}$$

where

$$\begin{aligned} F_3(\Delta R_4, \Delta X_4) &= f_1(\Delta R_4, \Delta X_4)\cos\varphi - f_2(\Delta R_4, \Delta X_4)\sin\varphi, \\ F_4(\Delta R_4, \Delta X_4) &= f_1(\Delta R_4, \Delta X_4)\sin\varphi + f_2(\Delta R_4, \Delta X_4)\cos\varphi, \end{aligned}$$

$$f_1(\Delta R_4, \Delta X_4) = k_5' \frac{(R_3 + R_{40} + \Delta R_4)\Delta R_4 + (X_{40} + \Delta X_4)\Delta X_4}{(R_3 + R_{40} + \Delta R_4)^2 + (X_{40} + \Delta X_4)^2},$$
$$f_2(\Delta R_4, \Delta X_4) = k_5' \frac{(R_3 + R_{40} + \Delta R_4)\Delta X_4 - (X_{40} + \Delta X_4)\Delta R_2}{(R_3 + R_{40} + \Delta R_4)^2 + (X_{40} + \Delta X_4)^2}, \tag{14}$$
$$k_5' = k_1' k_2 k_3 k_4, \quad k_1' = -I_{10} R_2.$$

Putting (13) as a normal system of equations, with notation as below, we have

$$\Delta R_4 = x_1, \ \Delta X_4 = x_2,$$
$$\frac{dx_1}{dt} = x_3, \quad T\frac{dx_3}{dt} = -x_3 + F_3(x_1, x_2),$$
$$\frac{dx_2}{dt} = x_4, \quad T\frac{dx_4}{dt} = -x_4 + F_4(x_1, x_2). \tag{15}$$

Our interest is directed to the behavior with the initial conditions

$$x_1^0 = \Delta R_4^0, \ x_2^0 = \Delta X_4^0, \ x_3^0 = x_4^0 = 0. \tag{16}$$

We restrict consideration to cases where T can be assumed sufficiently small, as is true of some designs of automatic bridges.

Then, using the results of [10], we can replace (15) by the degenerate system

$$\frac{dx_1}{dt} = x_3, \quad -x_3 + F_3(x_1, x_2) = 0,$$
$$\frac{dx_2}{dt} = x_4, \quad -x_4 + F_4(x_1, x_2) = 0 \tag{17}$$

with the initial conditions

$$x_1^0 = \Delta R_4^0, \ x_2^0 = \Delta X_4^0. \tag{18}$$

In fact the right halves of (15) are bounded and continuous in the x_i ($i = 1, \ldots, 4$), and define the functions $x_3 = F_3(x_1, x_2)$, $x_4 = F_4(x_1, x_2)$, also continuous in the x_i. The equations $F'_i = -x_i + F_i = 0$ ($i = 1, 2$) define the stable root (since $\frac{\partial F'_i}{\partial x_i} = -1 < 0$, $i = 1,2$) and these points in the region covered by (16) are points internal to the region of influence of the stable root $F'_i = 0$ ($i = 1,2$). Hence, from [10] (p. 202), when $T \to 0$ the solution to (15) subject to (16) tends at the limit to the solution to the degenerate system (17) subject to (18), this limit being stable relative to the initial conditions of (16).

(17) can be solved, using Erugin's method [11], by introducing the complex function $\Delta\dot{Z}_4$ of the real variable t:

$$\Delta\dot{Z}_4(t) = \Delta R_4(t) + j\Delta X_4(t). \tag{19}$$

Multiplying the second equation of (13) by j and combining with the first, allowing also for (19), and then putting $T = 0$, we get the trace of (17) as a single complex equation:

$$\frac{d\Delta\dot{Z}_4}{dt} = k_5' \frac{\Delta\dot{Z}_4}{R_3 + \dot{Z}_{40} + \Delta\dot{Z}_4} e^{j\varphi}. \tag{20}$$

Since the RHS of (20) is an analytic function of $\Delta\dot{Z}_4$ for all possible values of the latter, integrating (20) and using (18), we get a solution to (20):

$$t = \frac{R_3 + \dot{Z}_{40}}{k_5' e^{i\varphi}} \ln \frac{\Delta\dot{Z}_4}{\Delta\dot{Z}_4^0} + \frac{1}{k_5' e^{j\varphi}}(\Delta\dot{Z}_4 - \Delta\dot{Z}_4^0), \tag{21}$$

where $\Delta\dot{Z}_4^0 = \Delta R_4^0 + j\Delta X_4^0$.

Putting (21) in exponential form we have

$$e^{\frac{t}{\dot{B}}} = \frac{\Delta \dot{Z}_4}{\Delta \dot{Z}_4^0} e^{\frac{\Delta \dot{Z}_4 - \Delta \dot{Z}_4^0}{R_3 + \dot{Z}_{40}}}, \tag{22}$$

where

$$\dot{B} = \frac{R_3 + \dot{Z}_{40}}{k_5' e^{j\varphi}}. \tag{22a}$$

(22) shows that the sole balance point when $\mathrm{Re}\left[\frac{1}{\dot{B}}\right] = a < 0$ is $\Delta \dot{Z}_4 = 0$ ($\Delta R_4 = 0$, $\Delta X_4 = 0$). To show this it is sufficient to note that at $t = \infty$ and $a < 0$ (22) is satisfied only when $|\Delta \dot{Z}_4| = 0$. For all $|\Delta \dot{Z}_4| \neq \infty$ this follows directly from the relationships for the moduli of (22)

$$e^{at} = \frac{|\Delta \dot{Z}_4|}{|\Delta \dot{Z}_{40}|} e^{\frac{|\Delta \dot{Z}_4|}{|R_3 + \dot{Z}_{40}|} \cos(\gamma - \alpha_0)} e^{-\mathrm{Re}\left[\frac{\Delta \dot{Z}_4^0}{R_3 + \dot{Z}_{40}}\right]}, \tag{23}$$

where

$$\gamma = \arg \Delta \dot{Z}_4, \; a = \frac{k_5'}{|R_3 + \dot{Z}_{40}|} \cos(\varphi - \alpha_0).$$

To prove that $\Delta \dot{Z}_4 = \infty$ is not also a root of (22) when $t = \infty$ ($a < 0$), consider the limit

$$\lim_{|\Delta \dot{Z}_4| \to \infty} \Delta \dot{Z}_4 e^{\frac{\Delta \dot{Z}_4}{R_3 + \dot{Z}_{40}}} = -(R_3 + \dot{Z}_{40}) \lim_{|\Delta \dot{Z}_4| \to \infty} e^{\frac{\Delta \dot{Z}_4}{R_3 + \dot{Z}_{40}}}. \tag{24}$$

The modulus of this limit is

$$\left| \lim_{\Delta Z_4 \to \infty} e^{\frac{\Delta \dot{Z}_4}{R_3 + \dot{Z}_{40}}} \right| = \lim_{|\Delta \dot{Z}_4| \to \infty} e^{\frac{|\Delta \dot{Z}_4|}{|R_3 + \dot{Z}_{40}|} \cos(\gamma - \alpha_0)}. \tag{25}$$

When $|\Delta \dot{Z}_4| \to \infty$, $\cos(\gamma - \alpha_0) > 0$. This is readily seen by considering $\dot{Z}_4$ in the complex plane (Fig. 6). Thus when $|\Delta \dot{Z}_4| \to \infty$ the RHS of (22) tends (in modulus) to infinity, i.e., $|\Delta \dot{Z}_4| = \infty$ is not a root of (22).

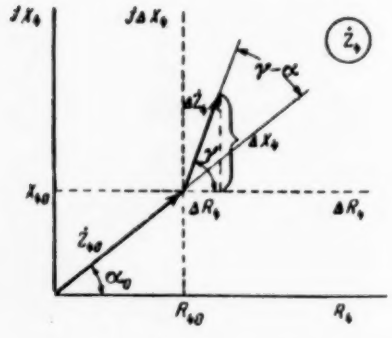

Fig. 6

Hence, when $a < 0$, any initial deviation $\Delta \dot{Z}_{40}$ eventually results in a return to balance. The condition $a < 0$, which amounts to the condition

$$\cos(\varphi - \alpha_0) > 0 \tag{26}$$

[since $k_5' < 0$; see (14)], is therefore the necessary and sufficient condition for stability "in the large" for this type of bridge.

Equations (13) were compiled for the balance point at which

$$\text{arc tan } \alpha_0 \frac{X_{40}}{R_3 + R_{40}}.$$

If the bridge is to be stable for all possible R_4 and X_4 we must have $\cos(\varphi - \alpha) > 0$, where the range in α is determined by the ranges in R_4 and X_4. Since in any bridge $0 < |\alpha| < \frac{\pi}{2}$ we can always ensure stability for all possible R_4 and X_4 by suitable choice of φ.

Condition (26) for the stability "in the large" is the same as the condition for the linearized system.

Hence the static nonlinearity does not here lessen the stability.

Equation (21) enables us to determine either the maximum time to reach balance for a given φ, or the φ for which the maximum time to reach balance is least.

APPENDIX

Neglect of Bridge Circuit Processes

Each balance circuit in the bridge of Fig. 1 contains modulated sections (bridge M, preamplifier P, and detector N).

To be able to use the methods employed with ordinary circuits (unmodulated) we must first establish the equivalent transfer function in terms of the envelope for the modulated section. This section has two inputs and a lag in the modulator.

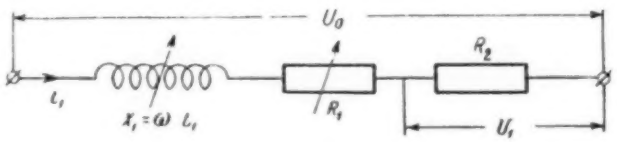

Fig. 7

Consider one bridge branch (Fig. 7), which is a modulator with a lag. We suppose the balancing element to be an inductance. The deductions below are equally correct if the reactive balance element is a capacitance.

The instantaneous branch parameters are given by

$$\frac{d}{dt}(L_1 i_1) + R_1 i_1 + R_2 i_1 = U_0, \tag{27}$$

Linearizing (27) about the balance point ($R_1 = R_{10}$, $L_1 = L_{10}$, $i_1 = i_{10}$) and supposing that

$$i_{10} = I_{10} \sin \omega \ t, \tag{28}$$

we get the branch response equation as

$$(\tau p + 1)\Delta U_1 = -k\left(\Delta R_1 + \frac{1}{\omega} p \Delta X_1\right) \sin \omega \ t - k \Delta X_1 \cos \omega \ t, \tag{29}$$

where

$$\tau = \frac{L_{10}}{R_{10} + R_2} = \frac{1}{\omega} \frac{X_{10}}{R_{10} + R_2}, \qquad k = I_{10} \frac{R_2}{R_{10} + R_2}, \quad \Delta U_1 = \Delta i_1 R_2.$$

From (29) the branch can be considered as a modulator, whose output is the modulated carrier

$$U_M = \left[\Delta R_1(t) + \frac{1}{\omega} p \Delta X_1(t)\right] \sin \omega \ t + \Delta X_1(t) \cos \omega \ t = f_1(t) \sin \omega \ t + f_2(t) \cos \omega \ t,$$

in series with a linear link of transfer function

$$K(p) = -\frac{k}{\tau p + 1}. \tag{30}$$

Hence the problem amounts to that of passing a modulated carrier $U_M = f_1(t) \sin \omega t + f_2(t) \cos \omega t$ through a linear circuit. The complex transfer function method given in [12] is convenient to use here. From [12], we introduce two independent operators $\underline{j}$ and $\underline{i}$ such that $i^2 = j^2 = -1$, but $ij = ji \neq -1$. Operator $\underline{i}$ refers to the signal envelope and $\underline{j}$ to the carrier.

The output signal from the linear link, in complex form, is

$$U_M = f_1(t) + jf_2(t), \tag{31}$$

or after a Laplace transform

$$U_M(S) = F_1(S) + jF_2(S), \tag{32}$$

where

$$F_1(S) = L[f_1(t)] = \Delta R_1(S) + \frac{1}{\omega} S\Delta X_1(S), \quad F_2(S) = L[f_2(t)] = \Delta X_1(S).$$

The variable transform S (envelope) is only complex in the $\underline{i}$ plane.

The input signal to the linear link of transfer function K(S) is defined by

$$\begin{aligned} U_{\text{out}}(S) &= \Delta U_1(S) = K(S + j\omega)\, U_M(S) = \\ &= K(S + j\omega) \left\{ \left[\Delta R_1(S) + \frac{1}{\omega} S\Delta X_1(S) \right] + j\Delta X_1(S) \right\}. \end{aligned} \tag{33}$$

(33) plus (30) define both complex transfer functions for the envelope for the branch

$$W_R(S) = \frac{\Delta U_1(S)}{\Delta R_1(S)} = \frac{-k}{\tau(S + j\omega) + 1} = \frac{-k[(\tau S + 1) - j\tau\omega]}{(\tau S + 1)^2 + (\tau\omega)^2}, \tag{34}$$

$$W_X(S) = \frac{\Delta U_1(S)}{\Delta X_1(S)} = \frac{-k\left(\frac{1}{\omega} S + j\right)}{\tau(S + j\omega) + 1} = \frac{-k\left[\frac{1}{\omega} S(\tau S + 1) + \tau\omega + j\right]}{(\tau S + 1)^2 + (\tau\omega)^2}. \tag{34a}$$

The equation for this branch will be

$$\Delta U_1(S) = W_R(S)\,\Delta R_1(S) + W_X(S)\,\Delta X_1(S). \tag{35}$$

For the other branch we have an analogous equation:

$$\Delta U_2(S) = W_R(S)\,\Delta R_4(S) + W_X(S)\,\Delta X_4(S), \tag{36}$$

where $\Delta R_4(S)$ and $\Delta X_4(S)$ are referred quantities (see Section 2).

The voltage on the measurement detector diagonal will be

$$\Delta U(S) = \Delta U_2(S) - \Delta V_1(S) = W_R(S)[\Delta R_4(S) - \Delta R_1(S)] + W_X(S)[\Delta X_4(S) - \Delta X_1(S)].$$

The output voltages from the detectors are given by

$$\begin{aligned} \varepsilon_1(S) &= \text{Re}_j\,[\Delta U(S)\, k_2 e^{j\varphi}] = \\ &= W'_R(S)[\Delta R_4(S) - \Delta R_1(S)] + W'_X(S)[\Delta X_4(S) - \Delta X_1(S)], \\ \varepsilon_2(S) &= \text{Re}_j\,[\Delta U(S)\, k_2 e^{j0}] = \\ &= W^*_R(S)[\Delta R_4(S) - \Delta R_1(S)] + W^*_X(S)[\Delta X_4(S) - \Delta X_1(S)], \end{aligned} \tag{37}$$

where

$$\begin{aligned} W'_R(R) &= \text{Re}_j\,[W_R(S)\, k_2 e^{j\varphi}] = \frac{-kk_2[(\tau S + 1)\cos\varphi + \tau\omega \sin\varphi]}{(\tau S + 1)^2 + (\tau\omega)^2}, \\ W'_X(S) &= \text{Re}\, j\,[W_X(S)\, k_2 e^{j\varphi}] = \frac{-kk_2\left\{\left[\frac{1}{\omega_R} S(\tau S + 1) + \tau\omega\right]\cos\varphi - \sin\varphi\right\}}{(\tau S + 1)^2 + (\tau\omega)^2}. \end{aligned} \tag{38}$$

The expressions for $W''_R(S)$ and $W''_X(S)$ are obtained by replacing φ by θ in (38).

The functions $W'_R(S)$, $W'_X(S)$, $W''_R(S)$, and $W''_X(S)$ are equivalent transfer functions for the measuring part of the bridge, which is the modulated section.

Supplementing (37) with the servodrive equations

$$\Delta R_4(S) = K_1(S)\,\varepsilon_1(S), \qquad \Delta X_4(S) = K_2(S)\,\varepsilon_2(S),$$

we get a system of equations describing the automatic bridge:

$$\begin{aligned}\left[\frac{D(S)}{K_1(S)} + A'_R(S)\right]\Delta R_4(S) + A'_X(S)\,\Delta X_4(S) &= A'_R(S)\,\Delta R_1(S) + A'_X(S)\,\Delta X_1(S),\\ A''_R(S)\,\Delta R_4(S) + \left[\frac{D(S)}{K_2(S)} + A''_X(S)\right]\Delta X_4(S) &= A''_R(S)\,\Delta R_1(S) + A''_X(S)\,\Delta X_1(S),\end{aligned} \tag{39}$$

where

$$\begin{aligned}D(S) &= \tau^2 S^2 + 2\tau S + 1 + Q^2,\\ A'_R(S) &= -kk_2\,[(\tau S + 1)\cos\varphi + Q\sin\varphi],\\ A'_X(S) &= -kk_2\left\{\left[\frac{1}{\omega}S(\tau S + 1) + Q\right]\cos\varphi - \sin\varphi\right\}\quad (Q = \tau\omega).\end{aligned} \tag{40}$$

The expressions for $A''_R(S)$ and $A''_X(S)$ are derived from those for $A'_R(S)$ and $A'_X(S)$ by replacing φ by θ.

The characteristic equation for the system is put as

$$\frac{D^2(S)}{K_1(S)\,K_2(S)} = A'_R(S)\,\frac{D(S)}{K_2(S)} + A''_X(S)\,\frac{D(S)}{K_1(S)} + A'_R(S)\,A''_X(S) - A''_R(S)\,A'_X(S) = 0. \tag{41}$$

In most cases of designing ac bridges $R_{10} + R_2$ and X_1 are chosen of the same order, because comparable sensitivities in both quantities measured are required [3]. Hence

$$\tau = \frac{1}{\omega_H}\,\frac{X_{10}}{R_{10} + R_2} = \frac{1}{\omega_H}\,\frac{X_{40}}{R_{40} + R_3}$$

is of order $\frac{1}{\omega} = \mu$, i.e., even at 50 cps it is of order 0.01.

We now examine how the stability can be judged from the degenerate characteristic equation obtained from (41) by putting $\tau = 0$.

Here it is necessary [13] to show that an accessory equation is stable.

The structure of A(S) and D(S), which are polynomials of identical degrees in S and τ, show that the characteristic equation is of the form

$$\begin{aligned}&\tau^n\,[M_{00}S^N + M_{01}S^{N-1} + \ldots + M_{0N}] + \tau^{n-1}[M_{10}S^{N-1} + M_{11}S^{N-2} + \ldots + M_{1N-1}] +\\ &\qquad \ldots + [M_{n0}S^{N-n} + M_{n1}S^{N-n-1} + \ldots + M_{nN}] = 0.\end{aligned} \tag{42}$$

Hence, the appropriate accessory equation must be a first-order one, i.e.,

$$M_{00}q^n + M_{10}q^{n-1} + \ldots + M_{n0} = 0. \tag{43}$$

Suppose

$$K_1(S) = \frac{P_1(S)}{SQ_1(S)}, \quad K_2(S) = \frac{P_2(S)}{SQ_2(S)}, \tag{44}$$

where $P_1(S)$ and $P_2(S)$ are polynomials of degrees l_1 and l_2, and $Q_1(S)$ and $Q_2(S)$ are polynomials of degrees m_1 and m_2, with $l_1 \le m_1$ and $l_2 \le m_2$.

We then see that the coefficients to the highest powers of S for each power of τ in the characteristic equation, i.e., M_{00}, M_{10}, . . . , M_{n0} in (43), coincide with those for the corresponding degrees of τS in $D^2(S)$. Hence, the accessory equation is obtained by equating $D^2(S)$ (with τS replaced by q) to zero:

$$D^2(q) = (q^2 + 2q + 1 + Q^2)^2 = 0. \tag{45}$$

This equation always has a stable solution.

Hence the stability "in the small" for automatic bridges with astatic balancing circuits in which X_1 and $R_{10} + R_2$ are of the same order can be judged from the degenerate characteristic equation obtained from (41) by putting $\tau = 0$, i.e., we can neglect transients in the bridge circuit.

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FREQUENCY METHODS OF REMOTE CONTROL FOR DISTRIBUTED OBJECTS

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The analysis leads to the selection and development of a frequency method of selecting and controlling distributed objects using line control channels. The design principles for remote control devices using frequency relays with series-resonant circuits are considered.

The methods and techniques of remote control so far developed for power systems, aircraft, railroads, etc. have mainly been for lumped objects.

As better process control methods are introduced the need arises to deal with many small control points or isolated objects distributed over areas or along lines (e.g. in oil- and gas-production, in irrigation systems, large factories, mines, pipelines, etc.).

The number of such points dealt with from one center may be tens or hundreds, while the total number of control or telemetering signals is small, seldom more than 3-5.

As the lines commonly used in such systems are relatively lightly loaded, and the signal to noise ratio is high, efficiency of information transfer and electrical noise effects are not commonly of major importance. The main problems in such systems are reliability in apparatus and lines, plus simplicity, convenience, etc.

The apparatus at the central station is usually under continuous observation, while the remote objects (outstations) are often in inaccessible places or in closed boxes in the open, and are left unattended for long periods. Hence, these latter objects must be much the more reliable, and the reliability of the system as a whole is largely determined by that of the outstations.

Various selection methods are considered here; the frequency method of selection and control is chosen. The design principles for remote control devices using frequency relays with series-resonant circuits are considered. The choice of line structure for distributed objects has already been dealt with [1].

Choice of Selection Method

The reliability at the controlled objects mainly depends on the properties of the electromagnetic relays, tubes, gas discharge devices, contactors, etc., as these are relatively liable to break down.

In the devices to be considered no tubes or gas devices are used, so the most unreliable elements can be considered to be the relays. Hence, in choosing methods the number of relays involved should be of primary importance (other things being equal) neglecting for the moment the numbers and loadings of the contacts.

The uses of contactless relays for this purpose have as yet been little studied, so these are not considered here. Their use undoubtedly ought to be examined and extended.

In choosing a selection method one must also ensure identity in the control equipment at the outstations, since this facilitates manufacture and servicing.

Remote control and telemetering devices for lumped objects (power grids, railroads, underground railways, etc.) often use time-division with the signals (e.g. in devices of VRT, RVS, KRV, DVK, etc. type which use time selection markers [1]). Attempts have been made to use the same method with distributed objects. Here each

effector point must control a rack of equipment with a distributor adequate to carry the full load of the system, although each such point may contain only one or two of the tens or hundreds of such objects in the system. Complex selection devices are then needed.

Let us determine the number of distribution and selection relays per object using time division methods with varying selection systems. We only consider a single line with a total of N objects connected to it [2].

Using time division methods the number of distribution and selection relays per object, k, depends on the selection method; in general k = m + n, where n is the total number of pulses in the code and m the number of selecting pulses.

If we neglect economies made by using repeated relay operations during a single transmission, rectifiers, etc., the number of distribution and selection relays per object can be described by the following formulae.

1. Distributive nongrouped method (N = n, m = 1)

$$k = N + 1.$$

2. Distributive grouped method with q degrees of selection and total number of objects per group $N^{1/q}$ and m = q

$$k = q(\sqrt[q]{N+1}).$$

3. Combination selection, with k = n + m for one set; then n, m and N are related by

$$N = \frac{n!}{(n-m)!m!}.$$

4. Combination selection, with for all sets

$$k = \frac{\log(N+1)}{\log 2} + \frac{\sum_{m=1}^{n} mC_n^m}{N}.$$

5. Distributive grouped method with distributors located at the objects, and with a number of pulses in the code equal to the number of objects; the line sections are connected to the group point one after the other when the code is being transmitted [3], and each object has a different number of code pulses:

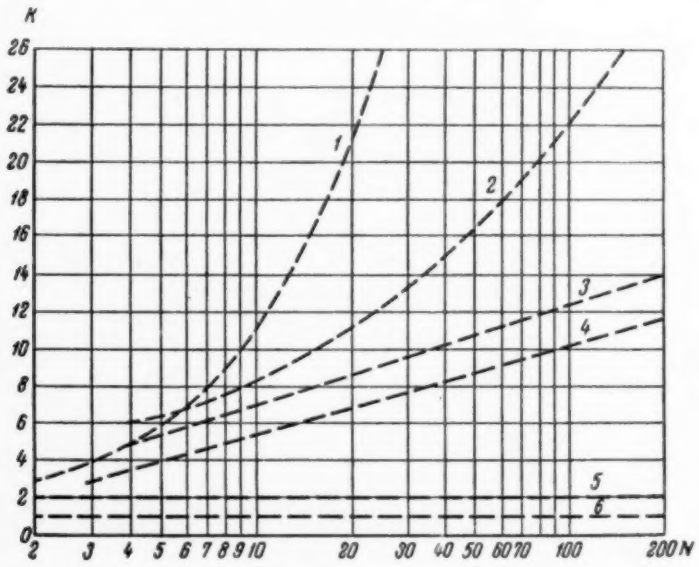

Fig. 1. Relation of the number of distribution and selection relays per object to the total number of objects with the various selection methods.

$$k = \frac{n}{N} + m = 1 + 1 = 2.$$

Figure 1 shows k = φ (N) for the various selection methods.

Figure 1 also shows the number of distribution and selection relays per object for the frequency method (curve 6, k = 1), each object having a distinct frequency. The relays are here the effectors, and play no part in selection.

Figure 1 shows that the number of distribution and selection relays per object can be minimal for the frequency method with distributed distributors [3].

All other methods require many more relays per object and (more important) the number of such relays depends on N.

Hence, using distributed distributors and frequency methods we can produce standard apparatus suitable for varying numbers of objects. Both methods have the substantial disadvantage that it is difficult to check the situation at the objects, particularly if the lines are long and the objects many.

The first method has the following disadvantages.

1. The lines and system generally are less reliable because they contain many contacts (at least N) connected in series. A fault in one relay puts the whole system out of commission. The structural reliability is low.

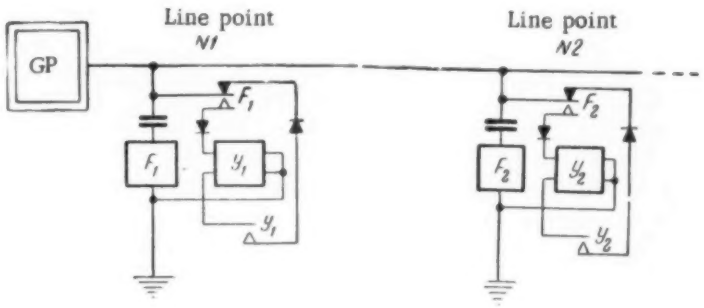

Fig. 2. Line point circuits using frequency selection and polarized control pulses.

2. It is difficult to prevent noise distorting the code, if a noise pulse transforms a counting pulse to a selection pulse. If all relays operate equally rapidly the selection time is some N times larger than in the frequency method. The probability of interference from noise is correspondingly increased.

3. It is difficult to ensure independent calls and telephone conversations using a single line, although this facility is in many cases necessary in such telemechanics systems.

These disadvantages restrict the use of the system. The frequency method also has disadvantages, the main one being that interference can produce distortions after selection (during the control or telemetering operations). This effect can be reduced to negligible proportions by using coding methods. For example, if the object is selected by frequency and polarized pulses used for control (e.g. for switching on the telemeter transducer) the relay can be arranged to be cut in by the pulses of one polarity only, and to be held on by current of reverse polarity. Figure 2 shows this system for a one-wire circuit. Here GP is the group point, the F_K (K = 1, 2) frequency selection relays, and the Y_K are control relays slugged on release. If a noise frequency occurs during the control time (after selection) and causes the frequency relay in some other object to operate (false selection), no undesirable or dangerous consequences arise, since a direct current is supplied to the line during the control period of sign opposed to that on which the control relay will operate.

A notable feature of frequency methods is that they are very flexible and that the various objects are virtually independent. If adequate margin is available in the frequency band, new objects can be connected without disturbing the old.

The frequency method is therefore often used with distributed objects, particularly if the line lengths are not more than a few tens of kilometers, or the number of elements per wire not over some dozens. The branched line structure being used make the use of this method more suitable.

Frequency Relays With Series Resonant Circuits

The frequency relay, consisting of relay plus frequency selection device, is the main component in frequency remote control and telemetering circuits.

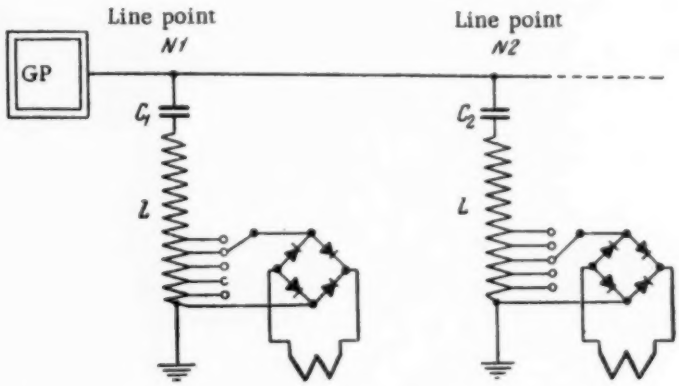

Fig. 3. Line points with series-resonant circuits and neutral relays.

Mechanical resonant systems (tuning forks, reeds, strings) electromagnetically driven have been much used for frequency selection; these either cause leaf spring contacts to close directly, or else operate polarized relays, magnetic amplifiers or transistors via rectifiers connected to their outputs. These systems have narrow response bands and are relatively stable [4-6]. A disadvantage is that the efficiencies are low and frequency resetting is difficult. Individual adjustment is required, the inventory has to be large, and the cost is high, as mass-production techniques cannot be used. Much better results are had by using series-resonant circuits with $Q = \frac{\omega L}{r_K} \gg 1$.

In the audio and subaudio ranges inductors built on standard toroid cores are best used. The toroid acts simultaneously as a stepdown autotransformer, with taps for selecting the operating voltage of the relay connected to it via a bridge rectifier (Fig. 3).

Figure 4 shows the experimental $Q = \varphi(f)$ curves (f = frequency) for coils wound with various wires on AlSiFe TCh-60 cores. The use of PEV wire instead of PELShO increases the Q by 25 to 40%. The impedance at resonance is usually made several times that of the line. The relation of load and resonant impedances is determined by the permissible reduction in the Q of the unloaded circuit. If the frequency is low and

$$\omega L_l < r_l \ll \frac{1}{\omega C_l},$$

where the subscript l's relate to the line, the simplified equivalent circuit for one such system connected to the line will be as in Fig. 5. Here r_i is the internal resistance of the source, $r_K = r_C + r_L$ is the resistance of the circuit at resonance and $\underline{r}$ the load resistance as seen by the circuit.

For an above-ground two-wire line 10 km long consisting of steel wire 4 mm in diameter operating at 200 cps we have

$$\omega L_l = 1256 \times 9 \times 10^{-2} = 110 \text{ ohm}, \quad r_l = 140 \text{ ohm},$$
$$\frac{1}{\omega C_l} = \frac{1}{1256 \times 10^{-2}} = 16 \text{ kohm}$$

The circuit can be designed to this line from

$$Q = \frac{\omega L_к}{r_к} = \varphi(f).$$

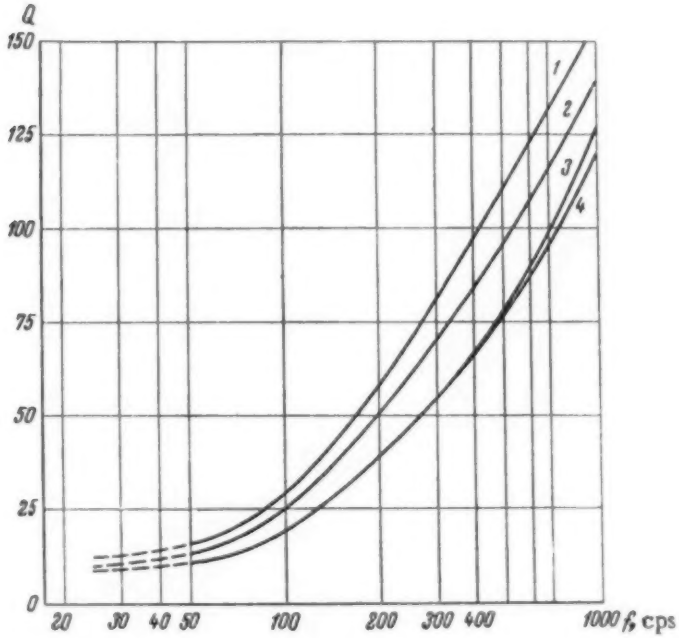

Fig. 4. Variation of Q with f for the toroids. Data on Toroids:

Curve No.	No. of turns	Type of wire	Wire diameter, mm	Inductance, henries,	Resistance, ohms
1	6500	PEV	0.31	6.15	125
2	6500	PEV	0.33	4.12	100
3	5500	PELShO	0.31	3.86	120
4	4300	PELShO	0.31	2.47	75

The efficiency at resonance with $\omega L \ll r_l$ for very low frequencies is given by

$$\eta_l = \frac{r}{r_i + r_l + r_к + r}.$$

In some cases it is desirable to use at very low frequencies the coefficients

$$k_1 = \frac{r_{l\ \min}}{r_{к\ \min}}, \quad k_2 = \frac{r_i}{r_{к\ \min}}, \quad k_3 = \frac{r}{r_{к\ \min}},$$

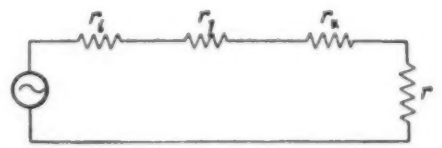

Fig. 5. Simplified equivalent circuit for one system connected to the line.

where $r_{l,\min}$ is the line resistance between the group point and the nearest object, and to assign the frequencies in such a way that the frequency increases with distance. In this way the reduction in efficiency at the higher frequencies can to some extent be compensated.

Being given k_1, k_2, and k_3, the efficiency and L_K can be determined from the $Q = \varphi(f)$.

The frequency relay resonance parameters determine the over-all Q:

$$Q_0 = \frac{\omega L_K}{r_i + r_l + r_K + r}.$$

r_K at the working frequencies is found from $Q = \varphi(f)$. Then $Q_0 = \varphi(f)$ curves are drawn up to determine the minimum spacing between frequencies, allowing for line parameters and supply voltage.

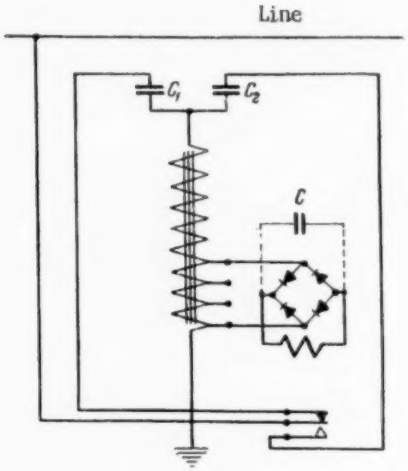

Fig. 6. Line point with a resonant circuit for selecting combinations of two frequencies.

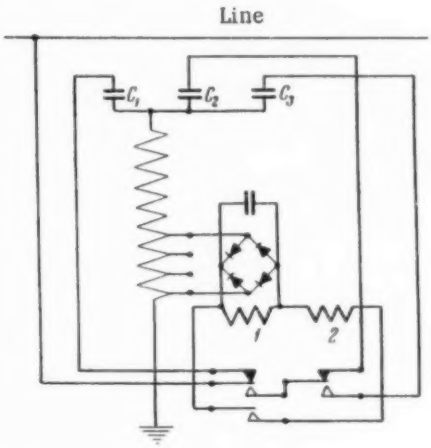

Fig. 7. Line point with a resonant circuit for selecting combinations of three frequencies.

Higher-resistance circuits are used for the longer lines, or with more numerous objects.

A Frequency-Combination Selection Method With One Relay Per Object

Electrically resonant relays enable one to use frequency-combination selection with one relay per object only; this method extends the use of the frequency method. When this new method is used with a group point in the line, two frequencies f_1 and f_2 are transmitted together or successively. f_1 is supplied briefly, and f_2 until the end of the operation. When f_1 arrives the relay, tuned to f_1 in the quiescent state, operates and switches the circuit over to f_2, by replacing C_1 by C_2, as in Fig. 6. f_2 holds the relay on throughout the operation.

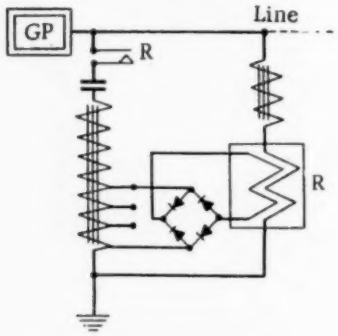

Fig. 8. Simplified telemechanics circuit using frequency selection and polarized signals.

The number of objects N (code formula) in this system will be $n_1 n_2$, where n_1 is the number of f_1's and n_2 the number of f_2's.

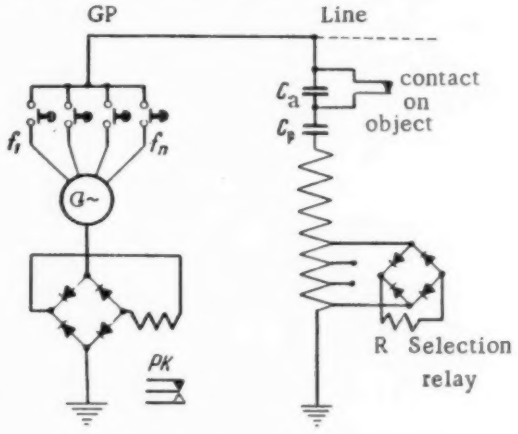

Fig. 9. Simplified telemechanics circuit using frequency selection and frequency interrogation of the position from the control.
C_p is a preset capacitor to tune to the working frequency, C_a an additional capacitor.

To increase the reliability the relay must be a slugged one, or the integrating capacitor C must be used. The discharge time-constant for C must be such that during the transfer period the voltage on C does not drop below the hold-on value.

This method can be used with three or more frequencies also. If three are used all the apparatus needed at the object is one inductor plus two electromagnetic relays 1 and 2, as shown in Fig. 7.

Frequency and polarized signals can be used simultaneously for selection. In this way circuits normally disconnected from the line can be used, as in Fig. 8. A brief dc pulse and the relevant frequency are dispatched together or successively from the group point.

The dc pulse passes through one winding of the relays R in all selectors, and causes them to operate. A relay can only hold on if its resonant frequency is also supplied. This greatly reduces the circuit loading of the line.

Remote Checking of Object Positions

Frequency relays using electrical resonance provide simpler means of checking object positions.

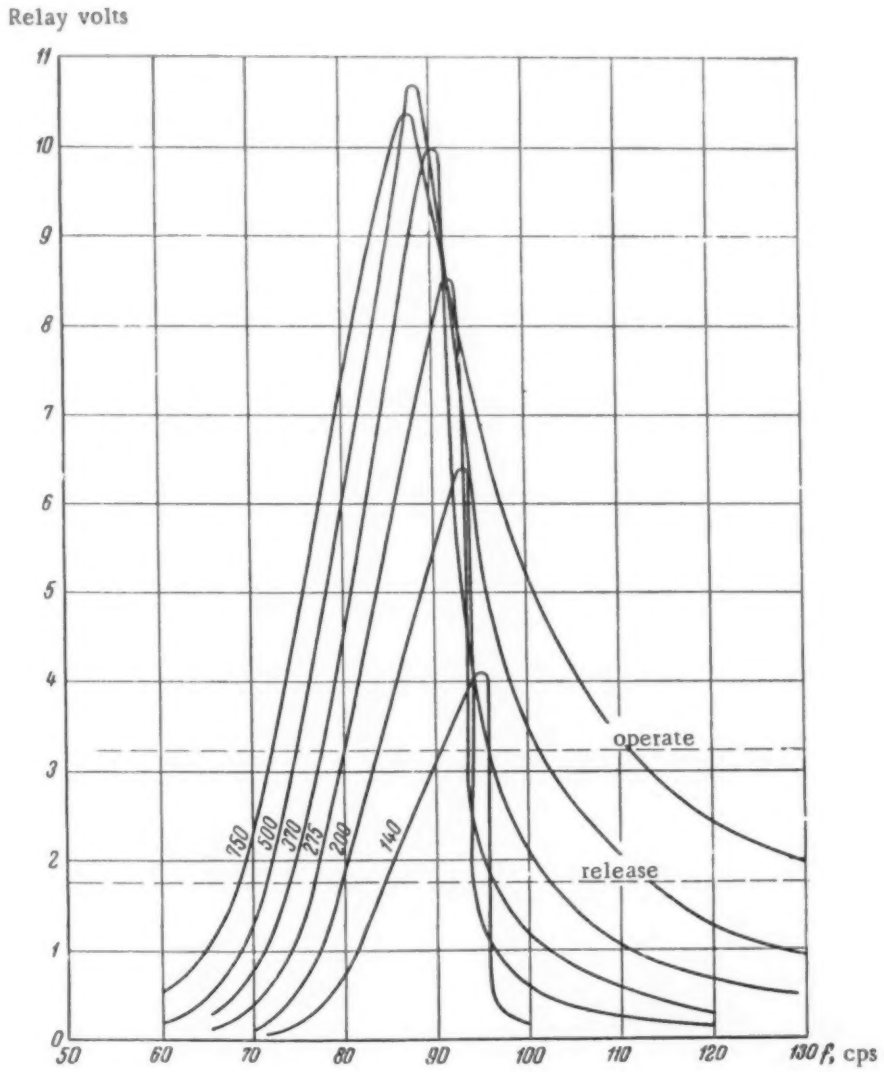

Fig. 10. Resonance curves for different toroidal coil windings connected to the load (relay). The numbers are those of the turns.

With short lines (up to 5-10 km) and $r_K > r_R$ the object position is best checked from the group point using the change in amplitude of the frequency used for selection. Figure 9 shows a circuit for this purpose.

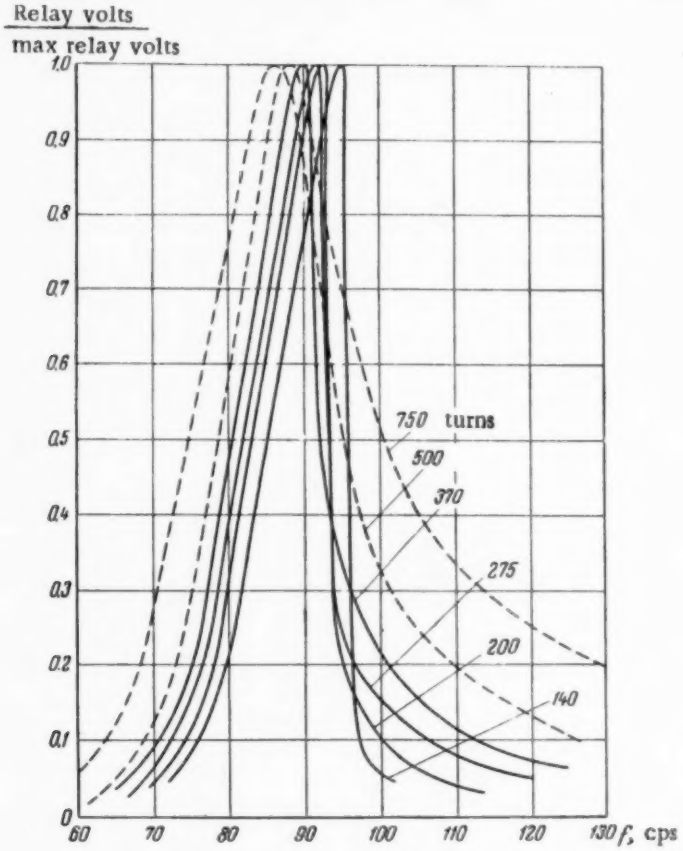

Fig. 11. Resonance curves for different toroidal coil windings connected to the load (relay).

TABLE

No. of curve group	Circuit capacitance, μf	No. of turns on secondary	Resonant frequency, cps	Line resistance to point, ohms		
				full curve	broken curve	dot and dash curve
1	9	500	25	0	—	—
2	4.5	500	34	0	—	—
3	2.8	500	44	0	—	—
4	2.27	370	54	0	11	15
5	1.32	370	71	0	14	24
6	0.74	275	92	0	17	29
7	0.45	275	113	0	21	36
8	0.31	200	137	0	24	43
9	0.24	200	157	0	27	50
10	0.17	200	182	0	30	57
11	0.138	200	210	0	33	64
12	0.106	140	240	0	36	71
13	0.08	140	269	0	39	78
14	0.063	140	300	0	42	85
15	0.05	140	335	0	45	92
16	0.039	140	373	0	48	—

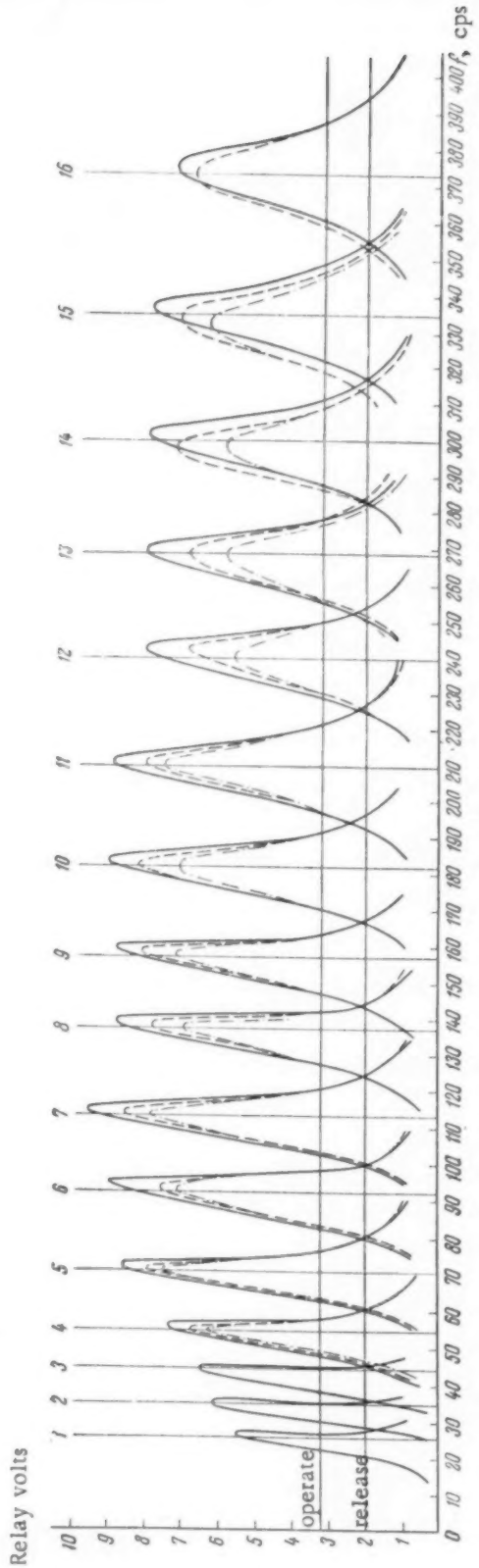

Fig. 12. Resonance curves for series circuits evenly spaced along the line.

The contact on the object switches an additional capacitor into or out of the circuit at the extreme positions, and thereby alters the resonant frequency. One frequency is the selection frequency, while the other can, for instance, be common to all objects, or all of a group. If the object is, for instance, switched on, the extra capacitor is cut out and the circuit has its proper frequency f_K. The resistance at resonance is reduced, and so the check relay PK at the central panel operates when f_K is transmitted. If the object is in the other position the extra capacitor is cut in, and the other frequency (e.g. f_0) is adopted; and so the check relay at the central panel does not operate when f_K is transmitted, since the impedance at this frequency is higher.

By transmitting a brief pulse of frequency f_0 and also f_K, the object can be cut in, and the switch-on checked. With lines over 5-10 km long this distant check is best done using the contact on the object to operate a local generator operating at the frequency proper to the point. This generator can be built round a transistor fed from a local battery or other source. The check can also be performed by using damped oscillations at the resonant frequency excited by a pulse sent from the control point.

EXPERIMENTAL RESULTS

Figure 10 shows resonance curves for a circuit as of Fig. 3. The relay voltage was measured at the output of a bridge rectifier connected to a RKN relay of resistance 350 ohms. The line voltage was constant at 20 volts. The toroid was wound with 0.31 mm diameter PELShO wire on an AlSiFe TCh-60 core. In all 5500 turns were wound, tapped at 750, 500, 370, 275, 200, and 140 turns. C = = 0.75 μ f. Curves were then taken for the load placed across various taps. Figure 11 shows the same curves on the relative scale (Relay volts/Max. relay volts) = φ (f).

Figures 10 and 11 show that as the resistance introduced by the load increases (with the number of turns) the resonance becomes flatter. The ferroresonance effect is also reduced, so the curves become less asymmetric. By using a working frequency somewhat below the resonant one, e.g. 90 cps, the ferroresonance effect can be neglected; this effect reduces the certainty of relay operation if the voltage alters.

Figures 10 and 11 show that as the damping increases (including effects due to increased line resistance) the method becomes of more limited value.

Figure 12 shows resonance curves for series circuits evenly spaced along the line.

The line and circuit parameters for each of the curves of Fig. 12 are given in the Table. In all cases the inductances were equal; the data have been given above.

The maximum resistance of the line from the generator to a circuit was 92 ohms, which corresponds to about 8 km of 4 mm diameter steel wire.

The demands as to the generator at the central point are comparatively easily satisfied with these circuits. Variations of 1-2% in frequency and 5-10% in output voltage are permissible. Harmonics are permissible up to 10-12%.

SUMMARY

1. The frequency method of selection results in least relays per object, which greatly reduces the total number of relays in systems with distributed objects. The remote control apparatus at the object can then be very simple, reliable and standard, in spite of variations in system size. This makes the method very suitable for systems with distributed objects.

2. Interference can cause a second object to be selected and controlled after a first has been selected in the frequency method. This basic disadvantage can be largely eliminated by using combined selection systems, e.g. frequency plus polarity or frequency combinations.

3. The LC serfes-resonant circuits described in the paper can often be used to advantage in systems with distributed objects as the frequency selection devices.

4. The frequency relay described can be used in frequency combination selection methods. One relay per object is then required.

5. The amplitude-frequency of checking the object position described above enables one to simplify the signalling system for lines up to 5-10 km long.

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• See English translation.

LETTERS TO THE EDITOR

A NONLINEAR SEMICONDUCTOR OF RESISTANCE SENSITIVE TO MAGNETIC FIELDS

G. E. Pilus and O. V. Sorokin

(Leningrad)

Several theoretical and experimental papers have been published on resistance changes in semiconductors in magnetic fields [1-6]. It has been shown that in semiconductors with carriers of two signs change in carrier concentration occurs on passing a current; this is additional to the normal increase in resistance (proportional to the square of the field) caused by the change in carrier mobility. If the current in the specimen and the external field are directed as shown in the figure and the specimen is a thin wafer of thickness d, then the total change in resistance will under certain conditions [1-6] be given by

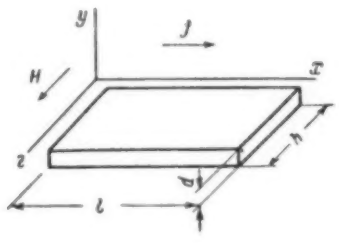
$$\frac{\Delta r}{r_0} = \frac{a_n}{4}\frac{\mu_n H}{c}(1+\gamma b)\frac{eU}{kT}\frac{d}{l}, \qquad (1)$$

where r_0 is the resistance at zero electric field, $b = \mu_p/\mu_n$, $\gamma = a_p/a_n$ (where μ_n and μ_p are the electron and hole drift mobilities and a_n and a_p are the ratios of Hall to drift mobilities for electrons and holes), U is the voltage applied, H the magnetic field strength, d the thickness, l the length, T the absolute temperature, k Boltzmann's constant, and e the absolute electronic charge.

Consider, for example, the characteristics of a nonlinear germanium resistor of inherent conductivity

$$n = p = 2.3 \times 10^{13}\ \text{cm}^{-3}, \quad \mu_n = 3700\ \text{cm}^2/\text{sec}, \quad \mu_p = 1700\ \text{cm}^2/\text{sec}, \quad a_n = 1, \quad \gamma = 1.2,$$

with a carrier diffusion length $\lambda_0 = 0.2$ cm and specific resistance $\rho = 50$ ohm·cm. We put $d = 0.15$ cm, $l = 1.5$ cm, $h = 0.5$ cm.



For room temperature (1) can be put as

$$\Delta r = 0.083 \times 10^{-3} r_0 UH, \qquad (2)$$

where Δr and r_0 are in ohms, U is in volts and H in oersted.

If, for instance, a permanent magnet with $H = 1200$ is used, then (2) gives

$$r = r_0(1 + 0.1\,U). \qquad (3)$$

A useful feature of this device is that the sign of the resistance change depends on the field directions. Since the resistors are small, a block of them, appropriately orientated and connected, can be placed together in the working gap of a single magnet.

The nonlinearity can be increased by replacing the permanent magnet by a magnet fed from the same source as the resistors. In this case the variation of Δr with H is also utilized. The resistance can also be adjusted by varying the magnetic field independently.

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CHRONICLE

THE INTERNATIONAL FEDERATION OF AUTOMATIC CONTROL (IFAC)

The Two First Periods of Development of IFAC

The International Congress of automatic control was held in Heidelberg (W. Germany) in September, 1956.

The congress was organized by the Control Group of the Union of German Engineers (VDI/VDE, Dusseldorf); the president was Dr. O. Grebe. Some 1000 representatives of 18 major states attended, including many famous engineers and scientists.

The soviet delegation, composed of A. M. Letov (Doctor of Physico-Mathematical Sciences), Ia. Z. Tsypkin (Doctor of Technical Sciences), V. N. Naumov (Candidate in Technical Sciences) and V. A. Racheev (Engineer) presented papers.

On September 27, Dr. Grebe called together representatives of the countries participating to discuss the proposal from Professor Oldenburger (U.S.A.) that an International Federation of Automatic Control be organized, with aims:

1) to facilitate the exchange of scientific information on automatic control between member countries, and

2) to organize international congresses on automatic control every four years.

The Soviet delegation gave its full support to this proposal, as extending contacts between scientists and engineers in different countries.

The aims of the federation were generally agreed, and the congress elected a Preparatory Committee composed of Professor V. Broide (France), Dr. Wellburn (Great Britain, president), Dr. O. Grebe (W. Germany), P. Nowatski (Poland), Professor Oldenburger (U.S.A.) and Dr. A. M. Letov (USSR).

The main task of the preparatory committee was to organize IFAC.

The work was divided into two sections:

1) establishing the views of countries on the federation;

2) developing proposals for the structure of the federation, for its statutes, and for calling the first General Assembly of IFAC.

In the USSR the first part was undertaken by an interim national committee set up by the Academy of Sciences.

The Preparatory Committee drew up a proposed charter for the federation, defining its aims, methods and means, membership, structure, rules and obligations on members, and also defining the functions of the General Assembly.

These statutes were drawn up using two alternative sets proposed respectively by Professors Broide (France) and Oldenburger (U.S.A.), and were adopted at a special session of the preparatory committee on April 25-27, 1957, in Dusseldorf. The resolution calling for the first General Assembly, and drafts of many resolutions to be presented at it, were also adopted.

The second meeting was set for September 9, 1957, in Paris, with the first General Assembly to follow on September 10-12 at the same place.

The section on membership occasioned the most discussion. The Soviet delegation considered that any country which expressed a desire to join IFAC should be admissible. Another view was that the membership question should be decided by vote at the next General Assembly. This was entered in the statutes.

In addition the committee agreed that states which notified their acceptance of the statutes of IFAC in writing prior to September 9, 1957 (i.e. prior to the first General Assembly) should be members.

Meeting of the Preparatory Committee September 9 and 10, 1957.

This meeting was held with Professor Broide in the chair; documents prepared by the general secretary (Dr. Ruppel, W. Germany) were first dealt with.

The most important of these dealt with amendments to the statutes considered at Dusseldorf, introduced by by the United Kingdom, U.S.A. and West Germany. That of the first was in the nature of a correction, whereas the two others were of a more radical character. The UK delegate proposed to extend the functions of the federation to computers, and to include "computation" in the title of the federation.

The Soviet delegate opposed this proposal, on the grounds that an independent computer federation already existed, and that workers in automatic control were only interested in computers insofar as these were component parts of control systems. All present agreed with this view.

The second proposal from the UK delegate amounted to including in the objects of the federation the study and development of technical automatic control apparatus. The Soviet delegate supported this proposal. Amendments by the U.S. delegate were of a very minor character, since the main proposals put forward earlier by Professor Oldenburger had already been agreed and endorsed. A very important extra section, dealing with the specialized committees of IFAC and with the rules governing their work, was proposed by the West German delegate.

On the other points the committee agreed to recommend to the General Assembly that:

1) subscriptions should be 125, 250, 500 and 1000 dollars per year. The subscription chosen was to be left to the honor of the member country, and would be published in the press;

2) Switzerland should be the permanent home of IFAC;

3) Dusseldorf (W. Germany) should be the permanent location of the secretariat.

On Professor Oldenburger's proposal, amended in the section on membership, the committee agreed to recommend to the General Assembly that the Executive Committee of IFAC should be composed of:

1) H. Chestnut (U.S.A.), president of IFAC;

2) A. M. Letov (USSR), first vice-president of IFAC;

3) V. Broide (France), second vice-president of IFAC;

4) H. Ruppel (W. Germany), secretary;

5) Lemann (France), treasurer;

6) P. Nowatski (Poland);

7) Gericke (Switzerland);

8) J. Coals (UK);

9) Tsien-Hsue-Sen (China);

10) Evangelisti (Italy);

11) M. Ainbinder (Belgium).

This list was later accepted at the General Assembly.

Finally, the committee decided to call the first International Congress of IFAC for 1960 in Moscow.

General Assembly of September 10, 11, and 12, 1957.

The formal opening of the meeting took place on September 10 in the evening. Representatives of the French scientific community welcomed the delegates. The working sessions opened on the following day (the 11th) with Professor Broide in the chair. A representative of the preparatory committee introduced all the delegates and observers. The assembly then considered the foundation statutes proposed by the committee, and the draft resolutions drawn up at its earlier meetings. All these resolutions were adopted.

Then the representative of the General Assembly, Professor Broide, formally declared IFAC in existence, and that the statutes of IFAC were submitted for ratification by all participating states, and by all others which had either sent observers, or which had indicated their desire to join IFAC prior to September 9. These states were Czechoslovakia, Yugoslavia, Chili, Finland, the German Democratic Republic and Denmark.

The First Meeting of the Executive Committee

The first meeting of the Executive Committee was held after the close of the General Assembly.

The following topics were discussed.

1. The structure of the first IFAC Congress.
2. Creation of advisory committees on scientific problems by the executive committee.
3. Time and place of the next meeting of the executive committee.

Professor Letov presented a preliminary view of the first topic. This was briefly as follows.

The Congress must cover three aspects of the scientific work of IFAC:

a) theory and methods of automatic control and regulation;

b) technical devices of automatic control and related problems;

c) industrial uses of automatic control and regulation (including the use of computer devices).

This view was subsequently expanded as to precise scientific content.

The first resolution on the structure of the Congress must be taken at its next session, set for March, 1958.

The International Federation of Automatic Control was thus created by resolution of its General Assembly.

The Soviet scientific community will undoubtedly welcome the creation of IFAC, and is now presented with an event of major scientific importance: the first international congress on automatic control in Moscow in 1960.

Specialists in all branches of automatics must begin now to prepare for this congress, in order to demonstrate the advanced level of theory and practice in automatic control and regulation in the USSR, and to prepare for showing to foreign scientists and engineers everything that is good and useful in our work.

A. M. Letov and B. N. Naumov.

CONFERENCE ON AUTOMATIC CONTROL AND COMPUTER TECHNIQUE

March 5-8, 1957, a Conference on Automatic Control and Computer Technique called by the All-Union Scientific Instrument Engineering Society was held in Moscow.

Nine hundred delegates from sections of the Academy of Sciences, and from the Academies of the Union Republics, higher teaching institutions and research institutes, design offices and laboratories of the Ministries and Councils participated.

Some 40 papers were read at the plenary and sectional meetings.

The introductory address was given by Rakovskii, who dealt with the general situation in automatic control and computer technique in the USSR and abroad and considered detailed examples of the work going on.

Much attention was devoted to problems related to cybernetics.

The first part of the address from Solodovnikov (Central Unified Automation Research Institute, as representative of the organizing committee) dealt with the tasks and program of the conference; the second dealt with defining the object and typical scientific content of technical cybernetics. He observed that the section of cybernetics concerned with theory and design principles of machines which effect algoritams can be termed technical cybernetics, and that technical cybernetics is composed of the following three fields:

1) information theory,

2) theory of logical and computing devices involving studies on methods of transforming information in accordance with set algorithms,

3) theory of automatic control, involving studies on methods of using information for effecting set purposes.

Liapunov demonstrated the importance of cybernetics as being the basis of unified automation. He stated that the theory of automatic control played the main part in cybernetics; this in its turn was determined by the nature and amount of the circulating information. The flux of circulating information increases with the complexity of the control system; optimal information coding problems arise.

This problem can be dealt with by using the coding methods employed in automatic translation; in particular, information should be stored in one code and handled in another. He also pointed out that multiple-step automatic control systems have advantages, and gave some examples indicating a need for studies on the control processes in living organisms.

The theory and design principles of control computers were dealt with in papers by Kazakevich, Batkov, Kuzin, Shreider, and by Solodovnikov, Batkov, Bredis, and Matveev in a joint paper.

Kazakevich dealt with "The principles and systems of extremum control," and presented the main concepts in the theory of extremum control. He remarked that these systems are fundamental to self-adjusting control systems, and gave several examples of peak-holding regulators.

The paper by Solodonikov, Batkov, Bredis, and Matveev (Central Unified Automation Research Institute) dealt with "The current state of the theory of optimal dynamical systems subject to random perturbations;" the importance of information theory to control system calculation and design was demonstrated, and the main ideas on which statistical methods of studying automatic control systems are based were reviewed.

Kuzin dealt with the Z-transform apparatus as used in digital computer automatic control system analysis and synthesis. He showed that these operations could thus be performed by methods analogous to those for analog computers. An example was considered of a thermal aircraft detector and follower system to illustrate the theoretical concepts.

Batkov dealt with a new application of electronic analogs for determining the correlation function and dispersion of the nonstandard random output from an automatic control system from the given characteristics of the input perturbation.

Shreider dealt with the design of "learning" machines, the main feature of which is that they can find the optimum control modes from accumulated experience. A necessary feature of such a system is that its behavior is to a certain extent undefined.

Dikushin, Liubimirskii, Kamynin and Shtarkman dealt with the development of the most logical automatic control systems for production processes, particularly those of metal-working machines.

Academician Dikushin (Metal-Cutting Machines Experimental Research Institute) dealt with the design of program controls, and indicated that proper choice of displacement and speed transducers for the effectors was essential. Displacement transducers must require the minimum memory volume as well as being as accurate as necessary. The proper form for the speed curves should be one of two steps; in the first the speed is large at large displacements, in the second it is less (as the set point is approached).

Dikushin's idea of "reflex control," the essence of which is that the effect is not controlled from the displacement direct, but from statistical observations, attracted much attention.

Liubimirskii, Kamynin, and Shtarkman's report (from the Steklov Mathematics Institute, Academy of Sciences of the USSR) dealt with the optimal coding of information in automatic control systems (including multiposition process controls). It is shown that for a lathe, for instance, doing some simple operation, no complex control was needed; the functions could be performed by a simple device. However, to supervise the functioning of the device a rather complex universal control machine is needed, but this can serve for several objects, and so the system is much simplified.

In conclusion Liubimirskii observed that the algorithms now available (e.g. for machine translation, for solving mathematical problems, etc.) are much more complex that those needed for many industrial processes; he pointed out that joint teams of mathematicians, technologists and computer specialists should be set up to solve process algorithm problems rapidly, and to design control computers.

The uses of high-speed computers are not limited to automatic process control or to mathematical problems. Computers can be used to process any information in any way, including translations from one language to another, programming programs, etc. and extensive development prospects are to be envisaged.

Shur-Bur's paper (from the Steklov Mathematics Institute, Academy of Sciences of the USSR) dealt with these topics.

A second extensive group of papers dealt mainly with the components and common features of computers, with programming methods, and with calculations and problems encountered in the automatic control field.

Recent advances in producing high-speed, reliable, cheap small computer components were dealt with by Korol'kov, by Mamonov and by Sharapov.

The first deal with some very promising high-speed magnetic switching elements using rectangular hysteresis loop materials. It was also pointed out that materials not having rectangular hysteresis loops could be used. Standard block units performing arithmetical, logical, etc. operations could be used in automatic devices.

The second dealt with semiconductor devices used in digital computers. The special features of point-contact and planar germanium transistors operating at large signals, and the various designs of trigger, were considered. Transistor triggers using carrier storage in planar diodes and the S-shaped response curves of point-contact diodes, circuits of units comprised of point-contact transistor dynamic triggers, or planar-contact static triggers (recorders, shift registers, read-out units) were presented.

Zimin's and Gutenmakher's papers dealt with the use of these components in computer circuits.

Gutenmakher showed that magnetic switching units can provide long-term and intermediate storage facilities, arithmetical, logical, etc. operations, or control and switching operations. Various circuits were illustrated for adding n-column binary numbers, for multipliers of series-parallel action for two n-column binary numbers, circulating registers, decimal recorders, switching devices, logical units, etc. all using magnetic switching units.

Zimin dealt with semiconductor devices used in computers, and pointed out that scientifically sound specifications for such devices should be developed.

Dealing with machine logic, he showed that all logical operations the machine can perform can be built up from the three basic elements "or," "and," and "no." These elements can be made to give low attenuation and satisfactory signal-to-noise ratio using diode plus transformer circuits. Registers and recorders using transistor amplifiers can be built up on these principles.

The input and output units of computers were considered by Ryzhov, by Trubnikov and by Zavolokin and Baskukov.

Ryzhov reviewed the foreign published data on analog-digital and digital-analog converters.

Trubnikov dealt with the comparative aspects of the input and output units of high-speed program-controlled computers, and analyzed the various forms of program carrier.

Zavolokin and Baskukov dealt with the input and output units of cyclic converters with inductive phase-shifters.

Val'denberg dealt with a specialized analog computer specially designed to solve the Fredholm and Volterra integral equations of first and second orders with nuclei of configuration type so often encountered in control problems.

The iterative Gauss-Seidel method was used. Digital elements were widely used. The machine found the solution periodically, without preserving the natural time-scale.

Mathematical statistics techniques can be used to deal with many physical phenomena. However, the great labor involved in computing many statistical parameters of random processes (particularly correlation functions) prevents their wide use in practice.

Novikov (Institute of Automation and Remote Control, Academy of Sciences of the USSR) has produced a specialized computer (a magnetic correlograph) for computing correlation functions automatically; the design of the device was described. The signal was recorded on a magnetic tape, covering the frequency range from dc up to 200 cps. The cycling time varied from 10 to 60 minutes.

Vitenberg described an automatic electronic analog for seeking the solution to a given system of equations. In relation to computers used as optimum solution seekers, he stated that electronic analogs were the best for use in such cases.

Survey and minimalization methods are the best for finding optimum solutions. An electronic analog for seeking the solution to a given system of equations consists of the analog proper plus control system and set of initial conditions, a timing unit, and a unit for controlling the logical operations of the input and output devices.

The comparative features and design methods of such devices were considered, and possible ways of production indicated.

Maiorov and Zhidkov dealt with the mathematical aspects of digital differential analyzers and with their use as control devices.

Zhidkov described how numbers were represented in digital differential analyzers and with ways of setting up the problems and choosing scales.

Maiorov dealt with how such analyzers are used in control systems covering many machines, how they are coupled to real objects, with their possible uses, and with how they compare with other digital computers.

Gutenmakher dealt with the prospects of using a new type of information and statistical machine in control systems. The rapid development of various types of store was pointed out, and the structures of future

types of information and statistical machine were considered in detail; it was pointed out that new types of reading device for such machines will take the lines of analogs to mental processes.

The applications of future types of information and statistical machine were considered (in chemistry, in statistics, in trade) to round off the paper.

Several papers dealt with computers designed to control actual industrial processes (those of Kopai-Gora, Khetagurov, Kaganov, Fitsner and Vasil'ev, Efroimovich and of Cheliustkin).

Efroimovich (Central Automatics Laboratory), Cheliustkin (Central Automatics Laboratory), and Kopai-Gora dealt with computers designed to control the main objects in ferrous metallurgy (blast-furnace processes, arc steel-melting furnaces, rolling-mills).

In all these papers the good prospects for using computers in ferrous metallurgy were pointed out, the state of the art abroad was summarized, and the situation in the USSR was shown to be such that the time was ripe for introducing computer control in this field. The lack of mathematical descriptions for the processes was indicated as greatly retarding this development.

Several examples of computer devices at present under test were mentioned. Much attention was devoted to future uses of computers. The 1956 All-Union Conference on Process Automation in Ferrous Metallurgy (in Magnitogorsk), at which a plan to provide full automatic working for several processes was adopted, was shown to have been very important.

The choice of most suitable cutting conditions is of great importance in increasing productivity in metal-working processes.

Vasil'ev and Fitsner dealt with a computer designed to handle this problem. Up to 20 distinct quantities which determine the cutting conditions can be fed in. If some of these are given, the required optimal parameters can be calculated in 2-3 minutes. Laboratory and works test results were stated to be satisfactory.

Khetagurov dealt with a digital control system for a metal-working machine.

After thorough discussions on the papers the Conference adopted several resolutions.

In view of the primary importance of fully automatic process control, it was resolved to extend research on the use of computers in automatic process control.

In doing this the main attention must be devoted to developing technical cybernetics theory, to producing new control machine components and standard units and racks, to developing new process techniques in various industries so that computer technique can be used.

It was also resolved that the Central Unified Automation Research Institute, the research institues, etc. of the Ministries and Councils, and the Academy of Sciences, should organize specialist groups on technical cybernetics. It was decided to ask the All-Union Scientific Engineering Society to hold a conference on cybernetics in collaboration with the Academy of Sciences in 1958.

This conference advanced the development of automatic control theory, made available experience with particular control problems involving computer techniques, and enabled specialists in different aspects to combine forces in order to solve the problems of full-scale process automation in the USSR.

I. M. Rusevich.

ERRATA

In the article by Ia. Z. Tsypkin in Volume 17, No. 12 (1956):

1. In Formula (2.13) and all following, $\Delta^k u_0[n]$ should be replaced by $(-1)^k \Delta^k u_0[n-k]$ $(k = 0, 1, 2, \ldots)$.

2. Formula (2.19) should read:

for $r = 1$

$$u_{in\,st}[n] = d_0 n - d_1,$$

for $r = 2$

$$u_{in\,st}[n] = d_0 n^2 - d_1(2n-1) + 2d_2.$$